Towards obtaining SEP via Orthogonality of Projection Matrix in massive MIMO Networks

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***Abstract*—Random projection (RP) based detectors achieve an asymptotically faster detection than linear low complexity massive MIMO detectors with a comparable error performance. To characterize the complexity *v/s* error performance trade-off, this work paves a way to obtain the exact symbol error probability (SEP) for RP-based detectors. Given the Rademacher or very sparse random projection (VSRP) distribution of the projection matrix, this work proves the columns of projection matrix to be orthogonal with high probability, as stated in the Lemmas III.1 and III.2. This leads to obtaining the distribution of the random projection detector statistic and thereby leads to obtaining an exact SEP for a projection matrix with orthogonal columns in massive MIMO communication networks.**

***Index Terms*—Random Projection, Orthogonality, Detection.**

I. Introduction

Rapid advancements in wireless technologies and futuristic paradigms like the Internet of Everything (IoE) have led to dramatic user applications like, augmented/virtual reality, smart homes, smart agriculture, autonomous navigation, and self-driving automotive, *etc.*, requiring to connect billions of devices that demand a pervasive, reliable wireless connectivity and infrastructure [1]. The massive multiple-input multiple-output (MIMO) and millimetre wave technologies, working in conjunction, can provided the radio solution to the seamless connectivity, ever-increasing high throughput and bandwidth requirements of next-generation wireless network applications [2]. However, the widespread adoption of massive MIMO technology still requires addressing pertinent issues. For instance, the uplink signal detection at the base station in massive MIMO systems is of high computational complexity due to the high dimensional received signal and uplink MIMO channel.

Over the last decade, several low complexity signal detectors based on Gauss-Seidel [3] , Newton-Iteration [4] and successive over relaxation [5] , have been proposed. These detectors reduce the detection complexity of the conventional zero-forcing (ZF) and minimum mean squared error (MMSE) detectors by replacing the matrix inversion with iterative or series-based approximations. However, all of these detectors have an asymptotic detection complexities same as ZF and MMSE. A class of low complexity random projection (RP)-based detectors, namely the Rademacher, Very Sparse Random Projection (VSRP), and Fast Johnson Lindenstrauss Transform detectors (FJLT), for massive MIMO networks were presented in [6] where the authors could present the bound on the symbol error probability (SEP) performance, but not the exact SEP expression.

We observe for RP based detectors in [6], exact SEP expression can be obtained when the columns of the projection matrix $\mathbf{P}$ are orthogonal. This work shows the columns of the projection matrix are orthogonal with high probability, presented in the Lemmas III.1 and III.2, for the RP-based detectors [6] in the massive MIMO communication networks. Given the orthogonal columns of projection matrix, the SEP performance can be obtained using the analysis for the signal in the additive white Gaussian noise.

II. System Model and Detection Framework

Consider an uplink massive MIMO scenario with U single antenna user terminals communicating with a large antenna array base station with B antennas. The real-equivalent of the complex base-band massive MIMO system model [6], considering small and large scale fading channel coefficient for $N = 2B$ and $M = 2U$, is given as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{1}$$

with the equivalent observations $\mathbf{y} \in \mathbb{R}^{N\times 1}$, the equivalent channel coefficient matrix $\mathbf{H} \in \mathbb{R}^{N\times M}$ and the equivalent white Gaussian noise vector $\mathbf{n}$, where $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma_n^2\mathbf{I}_N)$. The well known zero-forcing (ZF) detector aims to solve the cost function $\hat{\mathbf{x}}_{\text{ZF}} = \arg\min_{\mathbf{x}\in\mathbb{R}^M} \|\mathbf{y} - \mathbf{H}\mathbf{x}\|_2$, where $\|.\|_2$ denotes ℓ_2 norm, which yields the sufficient ZF statistic as

$$\hat{\mathbf{x}}_{\text{ZF}} = \mathbf{H}^{\dagger}\mathbf{y}, \tag{2}$$

where $(.)^{\dagger}$ denotes the Moore-Penrose pseudo-inverse. The complexity to obtain $\hat{\mathbf{x}}_{\text{ZF}}$ is $O(NM^2)$, which is very high for large values of N and M. This work presents a road map to characterize the exact performance of the random projection-based fast detectors, presented in [6], that achieve asymptotically faster detection performance compared to the existing linear detectors stated in the introduction section in the massive MIMO communication networks. The next subsection briefly states the necessary background for the random projection-based fast detectors.

A. Random Projection-based Detectors [6]

Johnson-Lindenstrauss transforms provide an efficient algorithm to obtain a low dimensional representation or "random projection (RP)" of a set of high dimensional data points while preserving their pairwise Euclidean distance with high probability [7]. The lemma is stated as follows:

Lemma II.1. ***Johnson and Lindenstrauss [7]:*** *Let ϵ, $\delta > 0$ be two parameters. Let $\mathbf{v} \in \mathcal{V} \subset \mathbb{R}^N$ such that $|\mathcal{V}| = M$. Then there exists a mapping $\mathbf{P} : \mathbb{R}^N \to \mathbb{R}^L$, where $L = O\left(\frac{\log M}{\epsilon^2} \log \frac{1}{\delta}\right)$, such that $\forall \mathbf{u}, \mathbf{v} \in \mathcal{V}$, following holds with probability at least $1 - \delta$,*

$$(1-\epsilon)\|\mathbf{u}-\mathbf{v}\|_2^2 \le \|\mathbf{Pu}-\mathbf{Pv}\|_2^2 \le (1+\epsilon)\|\mathbf{u}-\mathbf{v}\|_2^2.$$

The mapping can be taken as a matrix $\mathbf{P} = \frac{1}{\sqrt{L}}\mathbf{R}$, where $\mathbf{R} \in \mathbb{R}^{L \times N}$ with its elements $R_{ij} \sim \mathcal{N}(0,1)$. A few follow-up results [8] suggest improved construction of $\mathbf{P}$ and provide a faster algorithm with almost similar guarantee stated in Lemma II.1. In their result, the projection matrix $\mathbf{P} \in \mathbb{R}^{L \times N}$ has entries with the following distribution

$$\mathbf{P}_{ij} = \sqrt{\frac{s}{L}} \begin{cases} 1 & \text{with probability } \frac{1}{2s}, \\ 0 & \text{with probability } \frac{s-1}{s}, \\ -1 & \text{with probability } \frac{1}{2s}, \text{ for } s \ge 1. \end{cases} \quad (3)$$

For $s = 1$ and $s > 1$, the above distributions are referred to as Rademacher and VSRP distributions, respectively.

Let $\mathbf{P} \in \mathbb{R}^{L \times N}$, $L \ll N$ denote a random projection matrix that satisfies the guarantee stated in the Lemma II.1. The received signal, upon random projection with the matrix $\mathbf{P}$ is given as $\mathbf{w} = \mathbf{Py}$. Note the reduction in the size of the received signal vector $\mathbf{w} \in \mathbb{R}^L$, further using (1) we get

$$\mathbf{w} = \mathbf{P}(\mathbf{Hx}+\mathbf{n}) = \bar{\mathbf{H}}\mathbf{x} + \bar{\mathbf{n}}, \quad (4)$$

where $\bar{\mathbf{H}} = \mathbf{PH}$ and $\bar{\mathbf{n}} = \mathbf{Pn}$. Using the ZF detection framework in (4) the random projection based decoded symbol can be obtained from the cost function

$$\hat{\mathbf{x}}_{\text{P}} = \arg\min_{\mathbf{x}\in\mathbb{R}^M} \|\mathbf{w} - \bar{\mathbf{H}}\mathbf{x}\|_2, \quad (5)$$

to yield $\hat{\mathbf{x}}_{\text{P}} = \bar{\mathbf{H}}^\dagger \mathbf{w}$. Using the variants of the random projection matrix $\mathbf{P}$ in (4), the work [6] presents a class of fast random projection-based detectors such as Rademacher, Very Sparse Random Projection (VSRP), and Fast Johnson Lindenstrauss Transform detectors (FJLT), i.e., $\hat{\mathbf{x}}_{\text{RP-ZF}}$, $\hat{\mathbf{x}}_{\text{VSRP}}$ and $\hat{\mathbf{x}}_{\text{FJLT}}$, respectively.

III. Orthogonality of the columns of Matrix P

Recall from Equation (5) that the detected symbol has the following expression:

$$\hat{\mathbf{x}}_{\text{P}} = \bar{\mathbf{H}}^\dagger \mathbf{w} = (\mathbf{H}^T\mathbf{P}^T\mathbf{PH})^{-1}\mathbf{H}^T\mathbf{P}^T\mathbf{Py} = \mathbf{x} + \mathbf{v}, \quad (6)$$

where $\mathbf{v} = (\mathbf{H}^T\mathbf{P}^T\mathbf{PH})^{-1}\mathbf{H}^T\mathbf{P}^T\mathbf{Pn}$. The closed-form expression of exact SEP of the random projection-based fast detectors [6] in massive MIMO communication networks remains an open problem. As, the authors in [6] could only compute a theoretical bound on the instantaneous SEP for the random projection-based fast detectors. This work demonstrates a way to derive the exact SEP expression when $\mathbf{P}^T\mathbf{P} = k\mathbf{I}$. It is imperative to obtain the distribution of the equivalent noise vector $\mathbf{v}$ to derive the exact SEP expression for a given channel matrix $\mathbf{H}$ when averaged over the distribution of the projection matrix $\mathbf{P}$. However, when $\mathbf{P}^T\mathbf{P}$ becomes a diagonal matrix, the sufficient statistic $\hat{\mathbf{x}}_{\text{P}}$ in (6) will be similar to a scenario of the signal $\mathbf{x}$ transmitted in the additive white Gaussian noise vector $\mathbf{v}$, where the noise vector $\mathbf{v}$ follows a Gaussian distribution, i.e., $\mathbf{v} \sim \mathcal{N}\left(\mathbf{0}, \sigma_n^2(\mathbf{H}^T\mathbf{H})^{-1}\right)$ and yields a tractable and exact SEP expression. Therefore, this section derives the probability with which the matrix $\mathbf{P}^T\mathbf{P}$ is a diagonal matrix.

The entries of the gramian matrix $\mathbf{P}^T\mathbf{P}$ are obtained as $(\mathbf{P}^T\mathbf{P})_{ij} = \begin{cases} \|\mathbf{p}_i\|^2 & i = j \\ \langle \mathbf{p}_i, \mathbf{p}_j \rangle & i \ne j \end{cases}$, where $\langle .,. \rangle$ denotes inner product and $\mathbf{p}_i$ the ith column of $\mathbf{P}$. When $s = 1$ in Equation (3), the term $\|\mathbf{p}_i\|^2$ is equal to 1, and when $s \ge 1$, then expected value of $\|\mathbf{p}_i\|^2$ is equal to 1. What remains is to show that non-diagonal entries $\mathbf{P}$ are zero. This is proved in Lemma III.1 and Lemma III.2, which shows the non-diagonal entries of $\mathbf{P}$ are equal to zero with high probability, for $s = 1$ and $s \ge 1$, respectively.

Lemma III.1. *Two arbitrary $L-$ dimensional vectors $\mathbf{u}$ and $\mathbf{v}$, whose entries are sampled independently from Rademacher distribution, are orthogonal with probability*

$$P_R = \frac{1}{2^L}\frac{L!}{(\frac{L}{2})!(\frac{L}{2})!}. \quad (7)$$

Proof. The inner product between vectors $\mathbf{u}$ and $\mathbf{v}$ is given as $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^L u_i v_i$. Consider a random vector $\mathrm{z} = [z_1, z_2, \dots z_L]^T$, where $z_i = u_i v_i$, $i \in [1, 2, \dots L]$. From (3), it is clear that z_i are independent and take $z_i \in \{+1, -1\}$ with equal probabilities. Let $\mathbf{z} = [-1, \dots -1, 1, \dots 1]$ be an instance of z such that the first $\frac{L}{2}$ elements are -1 and the remaining elements are 1. Hence, for a given $\mathrm{z} = \mathbf{z}$ the probability of orthogonality, i.e., $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^L z_i = 0$, is obtained as

$$\mathrm{P}[\langle \mathbf{u}, \mathbf{v} \rangle = 0 | \mathbf{z}] = \prod_{i=1}^{\frac{L}{2}} \mathrm{P}[z_i = -1] \prod_{i=\frac{L}{2}+1}^{L} \mathrm{P}[z_i = 1] = 2^{-L}. \quad (8)$$

Above partitioning of $L-$termed sequence into two parts, each corresponding to equal 1 and -1, be done in $L!/\left((\frac{L}{2})!(\frac{L}{2})!\right)$ ways, where $k!$ denotes factorial k. Hence, the overall probability of vectors to be orthogonal is obtained considering the conditional probabilities (8) over all instances of z as

$$P_R = \mathrm{P}[\langle \mathbf{u}, \mathbf{v} \rangle = 0] = \frac{1}{2^L}\frac{L!}{(\frac{L}{2})!(\frac{L}{2})!}.$$

□

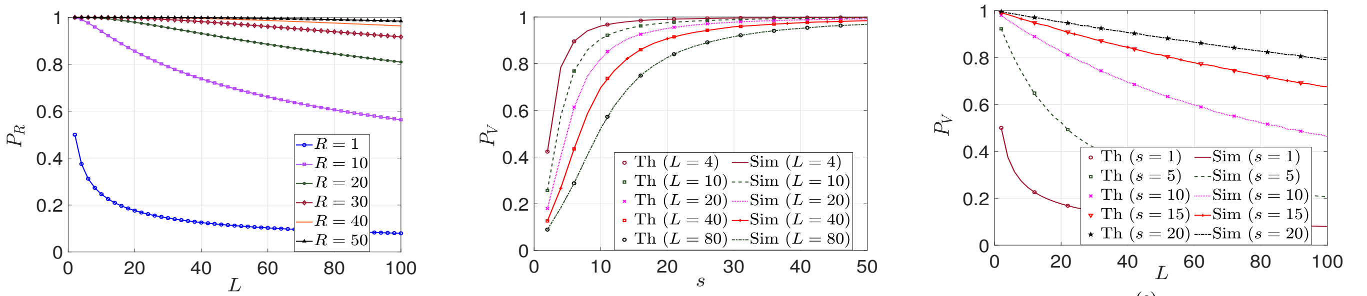
The following lemma summarizes the similar result for vectors sampled from VSRP distribution.

Lemma III.2. *Two arbitrary $L-$ dimensional vectors $\mathbf{u}$ and $\mathbf{v}$, with entries sampled independently from VSRP (sparsity parameter = s) are orthogonal with probability*

$$P_V = \left(\frac{1}{2s^2}\right)^L \sum_{i=0}^{\frac{L}{2}} \frac{L!}{((\frac{L-2i}{2})!)^2} \frac{(2(s^2-1))^{2i}}{(2i)!}. \quad (9)$$

Proof. The inner product of the vectors $\mathbf{u}$ and $\mathbf{v}$ be given as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^L u_i v_i = \sum_{i=1}^L z_i. \quad (10)$$



(a) (b) (c)

Fig. 1: For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^L$ the probability of orthogonality vs. (1a) vector dimension L for the Rademacher distribution where R denotes the number of repetitions to boost the probability of orthogonality from Remark III.3, (1b) sparsity s for VSRP with different dimensions L, and (1c) vector length L for VSRP with different sparsity s.

where $z_i = u_i v_i$, for $1 \leq i \leq L$, is a random variable with distribution $z_i = \begin{cases} +1 & \text{with probability } \frac{1}{2s^2} \\ -1 & \text{with probability } \frac{1}{2s^2} \\ 0 & \text{with probability } \frac{s^2-1}{s^2} \end{cases}$. Let $\mathrm{z} = [z_1, \cdots, z_i, \ldots z_L]^T$, then for the summation of z_i to be 0, there should be t_1 terms $= 0$, $\frac{t_2}{2}$ terms each equal to $+1$ and -1, where $t_1 + t_2 = L$. One such instance be where $t_1 = 2$ and $t_2 = L - 2$, denoted as $\mathrm{z} = \mathbf{z}$. Given that the partitioning of $L-$summation terms into the groups of 2, $\frac{L-2}{2}$ and $\frac{L-2}{2}$ can be done in $L!/\left((\frac{L-2}{2})!(\frac{L-2}{2})!2!\right)$ ways, the probability that the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal for instance $\mathrm{z} = \mathbf{z}$ be

$$\begin{aligned} \mathrm{P}[\langle \mathbf{u}, \mathbf{v} \rangle = 0 | \mathbf{z}] &= \frac{L!}{(\frac{L-2}{2})!(\frac{L-2}{2})!2!} \mathrm{P}[\mathrm{z} = \mathbf{z}] \\ &= \frac{L!}{(\frac{L-2}{2})!(\frac{L-2}{2})!} \left(\frac{1}{2s^2}\right)^{L-2} \left(\frac{s^2-1}{s^2}\right)^2. \quad (11) \end{aligned}$$

Adding the probability of all instances of z and algebraic simplification yields (9). □

Remark III.3. *Lemma III.1 gives the probability bound on one non-diagonal entry that takes value zero. The probability that all non-diagonal entries are zero is* $\tilde{P} = \left(\frac{1}{2^L} \frac{L!}{(\frac{L}{2})!(\frac{L}{2})!}\right)^{\binom{N}{2}}$ *where* $\binom{N}{2}$ *denotes the ways 2 objects can be chosen from N objects. The probability $\tilde{P}$ is further boosted to $1-\delta$, for $\delta > 0$, by repeating the experiments $O(\frac{1}{\tilde{P}} \log(\frac{1}{\delta}))$ times and picking the best solution. Analogous bound can also be given for VSRP distribution stated in Lemma III.2.*

Numerical Results: This paragraph discusses the numerical results, highlighting the trends in the orthogonality of random vectors. The Figures 1a, 1b, and 1c, presented elaborate the effect of varying the dimension (L) and sparsity parameter (s) on the orthogonality of random vectors. Figure 1a shows the variation of probability of orthogonality of two L dimensional vectors whose each entry is sampled from the Rademacher distribution. Note that with the increase in dimension, the probability of orthogonality reduces. Also, the boosting of probability due to repetitions, as stated in Remark III.3, can be observed as the increase in the number of repetitions (R) increases the probability of orthogonality.

Figure 1b and Figure 1c shows the variation of probability of orthogonality of two vectors from VSRP as a function of sparsity (s) and their dimension (L). The derived theoretical probability (denoted as 'Th') in III.2 match their simulation (denoted by 'Sim') counterparts. For a given dimension L, the probability of orthogonality increases with s. This agrees with the intuition that the randomly sampled sparse vectors with increased sparsity are more prone to being orthogonal.

IV. Conclusion and Future Work

This work presented a way to obtain an exact SEP expression for the RP based detectors in [6] for massive MIMO communication networks, by making the columns of the projection matrix $\mathbf{P}$ orthogonal. The result in the Lemmas III.1 and III.2, proved the columns of the projection matrix (obtained from Rademacher and VSRP distributions) are orthogonal with high probability. A similar observation was attained via simulations. Further analysis of SEP can be built upon the findings of this work. One of the ways would be to obtain the distribution of $\mathbf{v}$, when the columns of the projection matrix are orthogonal, to find the closed form SEP expression for the random projection-based detectors in massive MIMO communication networks.

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