Random Projection Based Efficient Detector in Massive MIMO Communication Networks

Gopal Chamarthi[‡], Adarsh Patel[‡], Rameshwar Pratap[♮]
[‡]School of Computing and Electrical Engineering (SCEE), Indian Institute of Technology Mandi, Mandi, India
[♮]Department of Computer Science Engineering, Indian Institute of Technology Hyderabad, Hyderabad, India
Email: UD22010@students.iitmandi.ac.in, adarsh@iitmandi.ac.in, rameshwar@cse.iith.ac.in

***Abstract*—This work introduces novel random projection-based efficient time complexity detectors for an uplink massive multiple-input multiple-output (MIMO) communication networks. The proposed Random projection-based detectors reduce the original dimension of the received symbols while preserving the pairwise Euclidean distance between the received and the corresponding transmitted symbols. Consequently, obtaining a faster detection algorithm with a comparable detection performance. Building on several variants of random projection such as Rademacher, Very Sparse Random Projection (VSRP), and Fast Johnson Lindenstrauss Transform detectors (FJLT), the corresponding detectors, $\hat{\mathbf{s}}_{\text{RP-ZF}}$, $\hat{\mathbf{s}}_{\text{VSRP}}$ and $\hat{\mathbf{s}}_{\text{FJLT}}$ are presented. A closed-form expression of the approximate symbol error probability (SEP) is obtained to characterize their performance. The time complexity of the proposed detectors is shown to significantly improve from the benchmark detectors, namely the maximum likelihood (ML), zero-forcing (ZF), and minimum mean-squared error (MMSE) detectors, along with the class of reduced complexity Neumann-series-based matrix inverse approximation (NS-MIA) detectors. The simulation results validate the tradeoff between the detection performance and the time complexity of the random projection-based detectors.**

***Index Terms*—Sketching, Random Projection based Detection.**

I. Introduction

Massive multiple-input multiple-output (MIMO) improves the reliability and efficiency of wireless communication systems and hence has become an integral part of the wireless standards [1]. However, these benefits come at the cost of increased hardware and complex signal processing algorithms [2]. For example, an uplink massive MIMO system with M single antenna users with a base-station (BS) having N antennas, the optimal Maximum Likelihood (ML) detector has an exponential time complexity. Similarly, zero-forcing (ZF) and minimum mean-squared error (MMSE) detectors achieve near ML detection performance [1] with a lower time complexity of $O(NM^2)$ [1]. However, for massive MIMO systems with large N, $N \gg M$, the time complexity of $O(NM^2)$ turns out computationally intensive [2]. To mitigate these challenges, a few fast and simplified detection techniques have been proposed to provide speedup at a compromised detection performance, and are discussed next.

Among the approximation/ iterative method based fast detectors, one such class of detectors reduces the complexity by approximating the inverse of the MIMO channel gram matrix which is of complexity $O(M^3)$. For the channel matrix $\mathbf{H} \in \mathbb{R}^{N\times M}$, the channel gram matrix is defined as $\mathbf{Q} = \mathbf{H}^H\mathbf{H}$. Detectors like Neumann-series based matrix inverse approximation (NS-MIA) [2], other variants like Gauss-Seidel (GS) [4], Newton-Iteration [5], Successive Over-Relaxation (SOR) [6] and quasi-newton based iterative detectors [7] aims to reduce the matrix inversion complexity from $O(M^3)$ to $O(M^2)$ [8]. The NS-MIA-based approximate detector achieves near ZF performance but converges under specific scenarios [9], [10]. The reduced gram matrix inversion complexity leads to the asymptotic overall detection complexity of $O(NM^2)$, which is computationally expensive for large values of N and M. Therefore, it is necessary to devise fast and simplified detection techniques that can provide processing speedup at a comparable detection performance.

Dimensionality reduction (*a.k.a.* sketching) techniques have been extensively used in various applications that involve high dimensional data by offering solutions of lower time and space complexity at the cost of tolerable error in performance. Johnson-Lindenstrauss [11] (*a.k.a.* random projection) is one of the classical dimensionality reduction algorithms for real-valued vectors. Their algorithm compresses high-dimensional real-valued vectors into low-dimensional such that compressed vectors closely approximates the original pairwise Euclidean distance. Further, several other dimensionality reduction algorithms are known depending on the data type and the underlying similarity measures [12]–[14]. These sketching algorithms have been widely used in problems belonging to linear algebra [15], data science/ machine learning [16] etc.

This work employs random projection [11] (Lemma II.1) based methods to achieve faster detectors. As the symbol error probability (SEP) performance depends on the minimum Euclidean distance between the symbols, the random projection-based techniques are a natural choice to provide faster detection algorithms as they preserve the pairwise Euclidean distances between the original and the compressed dimensions with high probability. Section IV proposes to obtain asymptotically fast detectors, namely $\hat{\mathbf{s}}_{\text{RP-ZF}}$, $\hat{\mathbf{s}}_{\text{VSRP}}$ and $\hat{\mathbf{s}}_{\text{FJLT}}$, by extending the advantages of several improved variants of random projection algorithms – Rademacher, Very Sparse Random Projection (VSRP), and Fast Johnson Lindenstrauss Transform detectors (FJLT), with a tolerable performance tradeoff in the massive MIMO wireless communication systems. Consequently, Section V derives a closed-form expression of SEP for the proposed detectors. Next, the work presents

This work was supported in part by the SCEE, IIT Mandi, Mandi, India.
[1]Notation $O(.)$ denotes asymptotic time complexity of the algorithm [3].

the time complexity analysis for the proposed and state-of-the-art detectors in Section VI. The simulation results presented in Section VII validate the proposed detectors time complexity and performance tradeoff. Matrices, vectors, and scalars are denoted using boldface capital, boldface small and regular letters, whereas the sets are denoted by calligraphic alphabets, *e.g.,* $\mathcal{S}$. The ℓ_2 norm of a vector $\mathbf{x}$ is denoted by $\|\mathbf{x}\|$.

II. Background

The seminal work of Johnson. W and Lindenstrauss suggest a dimensionality reduction algorithm for real-valued data such that the compressed vectors closely approximate the original pairwise Euclidean distance between the corresponding input vectors [11]. The lemma is stated as follows.

Lemma II.1. ***Johnson and Lindenstrauss (JL) [11]:*** *Let ϵ, $\delta > 0$ be two parameters. Let $\mathbf{v} \in \mathcal{V} \subset \mathbb{R}^N$ such that $|\mathcal{V}| = M$. Then there exists a mapping $\mathbf{T} : \mathbb{R}^N \to \mathbb{R}^K$, where $K = O\left(\frac{\log M}{\epsilon^2} \log \frac{1}{\delta}\right)$, such that $\forall \mathbf{u}, \mathbf{v} \in \mathcal{V}$, following holds with probability at least $1-\delta$, $(1-\epsilon)\|\mathbf{u}-\mathbf{v}\|_2^2 \le \|\mathbf{Tu}-\mathbf{Tv}\|_2^2 \le (1+\epsilon)\|\mathbf{u}-\mathbf{v}\|_2^2$, where $|\mathcal{V}|$ denotes the cardinality of set $\mathcal{V}$, ϵ the error tolerance, and δ the probability confidence.*

The mapping can be taken as a matrix $\mathbf{T} = \frac{1}{\sqrt{K}}\mathbf{R}$, where $\mathbf{R} \in \mathbb{R}^{K \times N}$ with its elements $R_{ij} \sim \mathcal{N}(0,1)$. A few follow-up results [17]–[19] suggest improved matrix construction $\mathbf{T}$ and provides a faster algorithm with almost same approximation in the pairwise Euclidean distance estimation.

III. System Description

Consider an uplink massive MIMO wireless communication scenario with F users, each having a transmit antenna, transmitting simultaneously to the BS with B receive antennas where $B \gg F$ [2]. The wireless channel matrix H is denoted as $\mathsf{H} = \mathsf{G}\boldsymbol{\Gamma} \in \mathbb{C}^{B \times F}$ between the users and the BS. The elements of $\mathsf{G} \in \mathbb{C}^{B \times F}$ are flat faded, follows a standard complex Normal distribution with zero mean and unit variance. The diagonal matrix $\boldsymbol{\Gamma} \in \mathbb{C}^{F \times F}$ has principle diagonal elements $\beta_f^{\frac{1}{2}}, 1 \le f \le F$, following a log-normal distribution with shadowing σ_s^2. The received signal $\mathsf{y} \in \mathbb{C}^{B \times 1}$ at the large antenna array BS corresponding to the transmission of F users signal $\mathsf{s} \in \mathbb{C}^{F \times 1}$ is expressed as

$$\mathsf{y} = \mathsf{Hs} + \mathsf{w}, \tag{1}$$

where the noise vector $\mathsf{w} \in \mathbb{C}^{B \times 1}$ is circularly symmetric and follows a Gaussian PDF $\mathsf{w} \sim \mathcal{CN}(\mathbf{0}, 2\sigma_w^2\mathbf{I}_B)$. The baseband system model in (1) can be equivalently written into a real-valued system model as

$$\underbrace{\begin{bmatrix}\mathfrak{Re}\{\mathsf{y}\}\\ \mathfrak{Im}\{\mathsf{y}\}\end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix}\mathfrak{Re}\{\mathsf{H}\} & -\mathfrak{Im}\{\mathsf{H}\}\\ \mathfrak{Im}\{\mathsf{H}\} & \mathfrak{Re}\{\mathsf{H}\}\end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix}\mathfrak{Re}\{\mathsf{s}\}\\ \mathfrak{Im}\{\mathsf{s}\}\end{bmatrix}}_{\mathbf{s}} + \underbrace{\begin{bmatrix}\mathfrak{Re}\{\mathsf{w}\}\\ \mathfrak{Im}\{\mathsf{w}\}\end{bmatrix}}_{\mathbf{w}}$$
$$\Rightarrow \mathbf{y} = \mathbf{Hs} + \mathbf{w}, \tag{2}$$

where the operators $\mathfrak{Re}\{\mathbf{y}\}$ and $\mathfrak{Im}\{\mathbf{y}\}$ denote real and imaginary parts of the complex vector $\mathbf{y}$. Assuming $N = 2B$ and $M = 2F$, the vectors $\mathbf{y} \in \mathbb{R}^{N \times 1}$, $\mathbf{s} \in \mathbb{R}^{M \times 1}$, $\mathbf{w} \sim \mathcal{N}(\mathbf{0}_{N \times 1}, \sigma_w^2\mathbf{I}_N)$ along with the massive MIMO channel coefficient matrix $\mathbf{H} \in \mathbb{R}^{N \times M}$. The transmit symbol $\mathbf{s}$ can take $S = |\mathcal{S}|$ values, where $|\mathcal{S}|$ denotes the cardinality of the set $\mathcal{S}$, corresponding to the considered signaling scheme, *i.e.*, $\mathbf{s} = \{\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_S\} \in \mathcal{S}$. For the massive MIMO system model described in (2) the optimal ML detector is presented next. The ML detector maximizes the likelihood $p(\mathbf{y}|\mathbf{s})$ which equivalently minimizes the symbol error, given as

$$\hat{\mathbf{s}}_{\text{ML}} \triangleq \arg\max_{\mathbf{s} \in \mathcal{S}}\ p(\mathbf{y}|\mathbf{s}) = \arg\min_{\mathbf{s} \in \mathcal{S}} \|\mathbf{y} - \mathbf{Hs}\|^2. \tag{3}$$

The ML detector is optimal in terms of SEP performance but has a higher computational complexity. The computation complexity grows exponentially with the size of the input signal vector $\mathbf{s}$, *i.e.*, M. The ZF relaxes the optimization problem in (3) to reduce the exponential computation complexity to $O(NM^2)$ by trading the SEP performance, given as

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s} \in \mathbb{R}^{M \times 1}} \|\mathbf{y} - \mathbf{Hs}\|^2. \tag{4}$$

The solution of optimization problem in (4) is obtained as

$$\hat{\mathbf{s}}_{\text{ZF}} = \mathfrak{F}\left(\mathbf{H}^\dagger\mathbf{y}\right), \tag{5}$$

where the operator $\mathfrak{F}(\mathbf{a})$ represent element-wise thresholding operation over the vector $\mathbf{a}$ and $(\cdot)^\dagger$ the Pseudoinverse. Next section proposes a random projection based detectors.

IV. Random Projection based Detector

The relaxed linear detectors, ZF in (5) and the MMSE in [1, (23)], have a time complexity of the order $O(NM^2)$. The SEP performance is proportional to the minimum pairwise Euclidean distance between the symbols. The random projection technique, proposed by JL in Lemma II.1, suggests transforming the $N-$dimensional vectors into a $K-$dimensional space for $K \ll N$ such that pairwise Euclidean distance is closely approximated, with a high probability. Hence random projection based algorithm becomes a natural choice to obtain faster detectors. The received vector $\mathbf{y} \in \mathbb{R}^{N \times 1}$ in (2) is transformed using the random projection matrix $\mathbf{T} \in \mathbb{R}^{K \times N}$ in order to obtain the compressed vector $\mathbf{z} \in \mathbb{R}^{K \times 1}$, given as

$$\mathbf{z} = \mathbf{Ty} \tag{6}$$

where $K < N$. The JL transformation (JLT) matrix $\mathbf{T}$ encodes the $N-$ dimensional vector $\mathbf{y}$ to a $K-$ dimensional vector, where $K = O\left(\frac{\log M}{\epsilon^2} \log \frac{1}{\delta}\right)$, with a guarantee to preserve $(1-\epsilon)\|\mathbf{y}\|_2 \le \|\mathbf{z}\|_2 \le (1+\epsilon)\|\mathbf{y}\|_2$ within $\epsilon-$ approximation [20]. Different JLT constructs were developed, defined as

$$\mathbf{T} = \begin{cases} \frac{1}{\sqrt{K}}\mathbf{R}_1 & \text{Rademacher} \\ \sqrt{\frac{s}{K}}\mathbf{R} & \text{Very Sparse Random Projections (VSRP)} \\ \mathbf{PUD} & \text{Fast JL transform (FJLT)} \end{cases} \tag{7}$$

where the ith row and the jth column element R_{ij} of the matrix $\mathbf{R}$ for VSRP [18] can take $\{1, 0, -1\}$ with probability $\{\frac{1}{2s}, \frac{s-1}{s}, \frac{1}{2s}$ for $s \ge 1\}$, respectively. For the sparsity parameter $s = 1$, it is called Rademacher distribution, *i.e.*, $\mathbf{R}_1 = \mathbf{R}$

at $s = 1$, and was proposed in [17]. Ailon and Chazelle [19] suggest FJLT by making the projection step faster, to employ

$$\mathbf{T} = \mathbf{PUD}, \tag{8}$$

where $\mathbf{D} \in \mathbb{R}^{N\times N}$ is a diagonal matrix whose diagonal entries are from Rademacher distribution. $\mathbf{U} \in \mathbb{R}^{N\times N}$ is a normalized Hadamard matrix. $\mathbf{P} \in \mathbb{R}^{K\times N}$ is a sparse matrix such that its elements $P_{ij} \sim \mathcal{N}(0, q^{-1})$ with probability q and $P_{ij} = 0$ with probability $1 - q$ where $q = \min\left\{\Theta\left(\frac{(\log M)^2}{N}\right), 1\right\}$. The notation $\Theta(.)$ denotes the asymptotic complexity of an algorithm/ operation [3]. The received signal $\mathbf{z}$ (6) in reduced dimension be solved using the JLT constructs in (7) to

$$\mathbf{z} = \mathbf{T}(\mathbf{Hs} + \mathbf{w}) = \bar{\mathbf{H}}\mathbf{s} + \bar{\mathbf{w}}, \tag{9}$$

where $\bar{\mathbf{H}} = \mathbf{TH} \in \mathbb{R}^{K\times M}$ and $\bar{\mathbf{w}} = \mathbf{Tw} \in \mathbb{R}^{K\times 1}$. Using the framework (3) to system (9), the random projection based ML detector for massive MIMO uplink communication is given as

$$\mathbf{s}_{\text{RP-ML}} = \arg\min_{\mathbf{s}\in\mathcal{S}} \|\mathbf{z} - \bar{\mathbf{H}}\mathbf{s}\|^2. \tag{10}$$

The relaxed optimization problem (4) when employing random projection (9) on to the received signal equivalently yields

$$\mathbf{s}_{\text{T}} = \arg\min_{\mathbf{s}\in\mathbb{R}^{M\times 1}} \|\mathbf{z} - \bar{\mathbf{H}}\mathbf{s}\|^2 \tag{11}$$

$$= (\mathbf{H}^T\mathbf{T}^T\mathbf{T}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{T}^T\mathbf{T}\mathbf{y}, \tag{12}$$

where the thresholding operator $\mathfrak{F}(\mathbf{s}_{\text{T}})$ is applied to obtain the equivalent $\hat{\mathbf{s}}_{\text{T}} = \mathfrak{F}(\mathbf{s}_{\text{T}}) \in \mathcal{S}$. The detected symbol $\hat{\mathbf{s}}_{\text{T}}$ when using the random projection matrix $\mathbf{T}$ in (7) for the Rademacher, VSRP and FJLT yields the random projection based detectors, $\hat{\mathbf{s}}_{\text{RP-ZF}}$, $\hat{\mathbf{s}}_{\text{VSRP}}$ and $\hat{\mathbf{s}}_{\text{FJLT}}$, for the uplink massive MIMO wireless communication networks. The detection performance of the proposed detector $\mathbf{s}_{\text{RP-ML}}$ is presented next.

V. SEP of Random Projection Based Detector

The analysis is presented for known channel coefficients matrix $\mathbf{H}$, random projection matrix $\mathbf{T}$, and the source symbols $\mathbf{s} = \{\mathbf{s}_1, \ldots, \mathbf{s}_S\} \in \mathcal{S}$, for any signaling scheme.

Theorem V.1. *The approximate SEP, i.e., the exact decision regions considering all the symbols is approximated by a tractable two-part decision regions obtained when considering two symbols at a time, considering the symbols* $\mathbf{s} = \{\mathbf{s}_1, \ldots, \mathbf{s}_S\} \in \mathcal{S}$ *for the proposed random projection based detector* $\mathbf{s}_{\text{RP-ML}}$ *in* (10) *for the massive MIMO uplink communication, with* $\mathbf{C}$=$(\mathbf{TH})^\dagger\mathbf{T}\mathbf{T}^T((\mathbf{TH})^\dagger)^T$, *obtained as*

$$P_e \leq \frac{1}{S}\sum_{j=1}^{S}\sum_{i=1, i\neq j}^{S} Q\left(\frac{(\mathbf{s}_i - \mathbf{s}_j)^T\mathbf{C}^{-1}(\mathbf{s}_i - \mathbf{s}_j)}{2\sqrt{\sigma_w^2(\mathbf{s}_i - \mathbf{s}_j)^T\mathbf{C}^{-1}(\mathbf{s}_i - \mathbf{s}_j)}}\right). \tag{13}$$

Proof. The closed form expression for the random projection-based detector in (12) using (2) is expressed as

$$\mathbf{s}_{\text{T}} = \bar{\mathbf{H}}^\dagger\mathbf{z} = (\mathbf{H}^T\mathbf{T}^T\mathbf{T}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{T}^T\mathbf{T}(\mathbf{Hs} + \mathbf{w})$$

$$= \mathbf{s} + \bar{\mathbf{H}}^\dagger\bar{\mathbf{w}}, \tag{14}$$

where the equivalent noise vector $\mathbf{n}$ is defined as $\mathbf{n} \triangleq \bar{\mathbf{H}}^\dagger\bar{\mathbf{w}}$ and follows a Gaussian density with zero mean vector and the covariance matrix $\mathbb{E}\{\mathbf{n}\mathbf{n}^T\} = \sigma_w^2\mathbf{C}$, where $\mathbf{C} = (\mathbf{TH})^\dagger\mathbf{T}\mathbf{T}^T((\mathbf{TH})^\dagger)^T$. Let S be the total number of symbols $\mathbf{s} \in \mathcal{S}$. Without loss of generality, let the symbol $\mathbf{s}_j, 1 \leq i, j \leq S$ was transmitted. The error event occurs when the detected symbol is $\mathbf{s}_i$, $i \neq j$. The probability of symbol error when $\mathbf{s}_j$ was transmitted can thus be written as

$$P_{e|\mathbf{s}_j} = P_{\mathbf{s}_j}(P_{\mathbf{s}_j\to\mathbf{s}_1} + \ldots + P_{\mathbf{s}_j\to\mathbf{s}_i} + \ldots P_{\mathbf{s}_j\to\mathbf{s}_S}).$$

$$= P_{\mathbf{s}_j}\sum_{i=1, i\neq j}^{S} P_{\mathbf{s}_j\to\mathbf{s}_i} \tag{15}$$

where, $P_{\mathbf{s}_j}$ denotes the probability of transmitting the jth symbol from the set $\mathcal{S}$, and $P_{\mathbf{s}_j\to\mathbf{s}_i}$ denotes probability of detecting symbol $\mathbf{s}_i$ when $\mathbf{s}_j$ was transmitted. Extending the same to all the symbols, the SEP can be expressed as

$$P_e = \sum_{j=1}^{S} P_{e|\mathbf{s}_j}, \tag{16}$$

where $P_{e|\mathbf{s}_j}$ is obtained in (15). Computing the exact symbol error using (16), *i.e.*, using the exact decision region corresponding to the transmission of the symbol $\mathbf{s} \in \mathcal{S}$ is challenging. To make the computations tractable, this work computes an approximate expression [21] for the probability of symbol error, and relaxes the computation of the exact decision region by the computation of an approximate decision region obtained considering two symbols at a time. When considering two symbols at a time, for instance $\mathbf{s}_i, \mathbf{s}_j$ for $1 \leq i, j \leq S$ and $i \neq j$, the symbol detection problem can be formulated as the binary hypothesis testing problem given as

$$\mathcal{H}_0 : \mathbf{s}_{\text{T}} = \mathbf{s}_i + \mathbf{n}$$

$$\mathcal{H}_1 : \mathbf{s}_{\text{T}} = \mathbf{s}_j + \mathbf{n}.$$

Hence, decoded symbol $\mathbf{s}_{\text{T}}$ corresponding to the transmission of the symbols $\mathbf{s}_i$ and $\mathbf{s}_j$ denoting the null $(\mathcal{H}_0)$ and the alternative $(\mathcal{H}_1)$ hypotheses, respectively. To decide in favour of the alternative hypothesis $\mathcal{H}_1$ the LRT [22] is given as

$$\mathcal{L}(\mathbf{s}_{\text{T}}) = \frac{p(\mathbf{s}_{\text{T}}|\mathcal{H}_1)}{p(\mathbf{s}_{\text{T}}|\mathcal{H}_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma', \tag{17}$$

where γ' denotes the decision threshold, and the two probability density functions (PDFs) are $p(\mathbf{s}_{\text{T}}|\mathcal{H}_0) \sim \mathcal{N}(\mathbf{s}_i, \sigma_w^2\mathbf{C})$ and $p(\mathbf{s}_{\text{T}}|\mathcal{H}_1) \sim \mathcal{N}(\mathbf{s}_j, \sigma_w^2\mathbf{C})$. Upon using these Gaussian PDFs in (17) and further solving the LRT in (17) the decision test statistic $T_{\text{T}}(\mathbf{s}_{\text{T}})$ equivalently can be obtained as

$$T_{\text{T}}(\mathbf{s}_{\text{T}}) = (\mathbf{s}_i - \mathbf{s}_j)^T\mathbf{C}^{-1}\mathbf{s}_{\text{T}} \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \gamma, \tag{18}$$

where $\gamma = \frac{1}{2}(\mathbf{s}_i - \mathbf{s}_j)^T\mathbf{C}^{-1}(\mathbf{s}_i + \mathbf{s}_j)$ is the decision threshold. When considering the transmitted symbol to be $\mathbf{s}_i$, among the two symbols $\mathbf{s}_i$ and $\mathbf{s}_j$, the probability of symbol error $\bar{P}_{\mathbf{s}_i\to\mathbf{s}_j}$ can be defined as,

$$\bar{P}_{\mathbf{s}_i\to\mathbf{s}_j} \triangleq P(\mathcal{H}_1|\mathcal{H}_0) = P(T_{\text{T}}(\mathbf{s}_{\text{T}}) \leq \gamma|\mathcal{H}_0) \tag{19}$$

$$= P\left((\mathbf{s}_i - \mathbf{s}_j)^T\mathbf{C}^{-1}(\mathbf{s}_i + \mathbf{w}) \leq \gamma\right). \tag{20}$$

Use (18) in (19) to get (20). The probabilities $P_{\mathbf{s}_i\to\mathbf{s}_j}$ in (15) and $\bar{P}_{\mathbf{s}_i\to\mathbf{s}_j}$ defined in (19), to confuse between the symbols $\mathbf{s}_j$ and $\mathbf{s}_i$, differ in the number of symbols under consideration. In $P_{\mathbf{s}_i\to\mathbf{s}_j}$, the symbol $\mathbf{s}_i$ can be confused with $\mathbf{s}_j, 1 \leq j \leq S, i \neq$

j whereas in $\bar{P}_{\mathbf{s}_i \to \mathbf{s}_j}$, the symbol $\mathbf{s}_i$ can only be confused with $\mathbf{s}_j, i \neq j$, i.e., the pairwise symbol error probability. The quantity $(\mathbf{s}_i - \mathbf{s}_j)^T \mathbf{C}^{-1}(\mathbf{s}_i + \mathbf{w})$ follows a Gaussian PDF which has a mean $\mu = (\mathbf{s}_i - \mathbf{s}_j)^T \mathbf{C}^{-1}\mathbf{s}_i$ and variance $\sigma^2 = \sigma_w^2(\mathbf{s}_i - \mathbf{s}_j)^T \mathbf{C}^{-1}(\mathbf{s}_i - \mathbf{s}_j)$. The SEP in (20) can now be expressed as

$$\bar{P}_{\mathbf{s}_i \to \mathbf{s}_j} = \mathrm{Q}\left(\frac{\mu - \gamma}{\sigma}\right) = \mathrm{Q}\left(\frac{(\mathbf{s}_i - \mathbf{s}_j)^T \mathbf{C}^{-1}(\mathbf{s}_i - \mathbf{s}_j)}{2\sqrt{\sigma_w^2 (\mathbf{s}_i - \mathbf{s}_j)^T \mathbf{C}^{-1}(\mathbf{s}_i - \mathbf{s}_j)}}\right).$$

Consider the symbols to be equally likely, $P_{\mathbf{s}_i} = \frac{1}{S}, \forall i$. Use the above probability $\bar{P}_{\mathbf{s}_i \to \mathbf{s}_j}$ in the overall SEP (16), a bound on the SEP is obtained (21). Further solving (21) yields (13).

$$P_e \leq \sum_{i=1}^{S} P_{\mathbf{s}_i} \sum_{j=1, i \neq j}^{S} \bar{P}_{\mathbf{s}_i \to \mathbf{s}_j}. \tag{21}$$

□

VI. Time Complexity Analysis of Random Projection based Detectors

The original detection problem mentioned in equation (3) is of exponential computational complexity. Its relaxed problem is listed in equation (4) is of polynomial time complexity $O(NM^2)$, where $N > M$. This is due to the computation of $\mathbf{H}^\dagger$, the pseudo-inverse of the channel matrix $\mathbf{H}$. Using random projection, the columns of $\mathbf{H}$, which are of $N-$ dimension, are embedded into a $K-$ dimensional space. The random projection based detectors involve the following two steps, (i) Projection of received signal, $\mathbf{z} = \mathbf{T}\mathbf{y}$. (ii) Computing $\hat{\mathbf{s}}_{\text{RP-ZF}}$ in (12) for the optimization problem in (11). The random projection matrix $\mathbf{T}$ follows the rademacher, VSRP, and FJLT distributions as defined in (7). For $\mathbf{T}$ from the Rademacher distribution, the random projection based detector computes the matrix $\bar{\mathbf{H}} = \mathbf{T}\mathbf{H}$, projection of received signal $\mathbf{T}\mathbf{y}$, and the solution to the optimization problem in (11), *i.e.*, the pseudo-inverse of $\bar{\mathbf{H}}$, which have time complexities $O(KNM)$, $O(KN)$ and $O(KM^2)$, respectively. Therefore, the overall time complexity to obtain $\hat{\mathbf{s}}_{\text{RP-ZF}}$, random projection based detector, is $\sim O(KNM + KN + KM^2)$, that is $O(KNM)$. The following theorem derives the computation complexity of other proposed detectors $\hat{\mathbf{s}}_{\text{VSRP}}$ and $\hat{\mathbf{s}}_{\text{FJLT}}$.

Theorem VI.1. *For a massive MIMO channel matrix $\mathbf{H} \in \mathbb{R}^{N \times M}$ defined in* (2) *and random projection matrix $\mathbf{T} \in \mathbb{R}^{K \times N}$ that satisfies JL Lemma in Lemma II.1, the random projection based detectors $\hat{\mathbf{s}}_{VSRP}$ and $\hat{\mathbf{s}}_{FJLT}$, obtained as a solution to the cost function in* (11)*, have the complexities*

- *$O\left(\frac{KNM}{s}\right)$, when $\mathbf{T}$ follows a Very Sparse Random Projection (VSRP) with sparsity parameter s, and*
- *$O(NM \log N)$, when $\mathbf{T}$ follows a Fast JLT (FJLT).*

Proof. For $\hat{\mathbf{s}}_{\text{VSRP}}$, consider the JLT constructed from VSRP [18]. In this setting $\mathbf{T} = \sqrt{\frac{s}{K}}\mathbf{R}$, the element R_{ij} of $\mathbf{R}$ can take the value $R_{ij} \in \{1, 0, -1\}$ with probability $\{\frac{1}{2s}, \frac{s-1}{s}, \frac{1}{2s}$ for $s \geq 1\}$, respectively. Therefore, the number of nonzero entries in the matrix $\mathbf{T}$ is $\frac{KN}{s}$, in expectation. As a consequence, the complexity of the projection step, that is the multiplication of $\mathbf{T} \in \mathbb{R}^{K \times N}$ and $\mathbf{H} \in \mathbb{R}^{N \times M}$ is $O\left(\frac{KNM}{s}\right)$, in expectation. For the second part, the projection matrix is considered from Fast JLT, that is, $\mathbf{T} = \mathbf{PUD}$ (8). Further, due to Lemma 2.1, [19] the time complexity of multiplication of $\mathbf{T} \in \mathbb{R}^{K \times N}$ and $\mathbf{H} \in \mathbb{R}^{N \times M}$ is $O(NM \log N)$. □

Remark VI.2. *The proposed random projection based detectors $\hat{\mathbf{s}}_{VSRP}$ and $\hat{\mathbf{s}}_{FJLT}$ are asymptotically faster than the ZF in* (5) *and the MMSE in [1, (23)] when $\frac{K}{s} = o(M)$ and $\log N = o(M)$, respectively, where $o()$ denotes* `small-oh` *notation of asymptotic time complexity analysis [3].*

The order complexities of different detection schemes for an $N \times M$ masive MIMO system are summarized in the TABLE I. The proposed RP-based detectors guarantee asymptotic faster detection from the ZF, MMSE and from the class of low complexity massive MIMO detectors in [23].

TABLE I
Time Complexity Comparison of Various Detectors

Detector	Time Complexity
ZF (Eq.(5)), MMSE [1, (23)], NS-MIA [10, (9)]	$O(NM^2)$
$\hat{\mathbf{s}}_{\text{RP-ZF}}$, proposed in Section IV	$O(KNM)$
$\hat{\mathbf{s}}_{\text{VSRP}}$, proposed in Section IV	$O\left(\frac{KNM}{s}\right)$
$\hat{\mathbf{s}}_{\text{FJLT}}$, proposed in Section IV	$O(NM \log N)$

VII. Simulation and Results

Considers an uplink massive MIMO scenario with $F = \{6, 30\}$ number of transmit antennas and $B = \{60, 128\}$ number of receive antennas at the BS. The wireless communication channel is considered to be Rayleigh flat faded, with parameter $\sigma_s^2 = 8$dB. The noise is assumed to be complex circularly symmetric, additive, white having a Gaussian density, *i.e.*, $\mathbf{w} \sim \mathcal{CN}(\mathbf{0}, 2\sigma_w^2 \mathbf{I}_B)$. The entries of the matrix $\mathbf{T}$ in (7) follow Rademacher, VSRP and FJLT distributions corresponding to the proposed detectors $\hat{\mathbf{s}}_{\text{RP-ZF}}$, $\hat{\mathbf{s}}_{\text{VSRP}}$ and $\hat{\mathbf{s}}_{\text{FJLT}}$, respectively. Without loss of generality, the elements of the transmitted vector $\mathbf{s} \in \mathbb{R}^{M \times 1}$ can take $s_i \in \{1, -1\}, 1 \leq i \leq M$. The compression ratio/ percentage indicate the ratio/ percentage between the reduced dimension K to the original dimension N. The proposed RP-based detectors being linear, their performance is compared with baseline *i.e.,* ZF and MMSE [1], and other low complexity linear detectors like NS-MIA [10], Gauss-Seidel [4], and SOR [6].

Fig. 1 presents the SEP vs. signal-to-noise ratio (SNR) performance comparison for a $60 \times 6-$MIMO with $\sigma_s^2 = 8$dB. A superior performance of the random projection based ML detector $\mathbf{s}_{\text{RP-ML}}$ (10) over the random projection based ZF detector $\hat{\mathbf{s}}_{\text{RP-ZF}}$ (12) is observed. The detection performance of the proposed random projection based detectors, namely $\mathbf{s}_{\text{RP-ML}}$ and $\hat{\mathbf{s}}_{\text{RP-ZF}}$, are close to the original dimension based ML and ZF detectors for a higher $\frac{K}{N}$. Also, the performance vs. time complexity trade-off for the proposed detectors can be observed, *i.e.*, the performance of the proposed detectors improves with an increase in $\frac{K}{N}$, for $\frac{K}{N} \in \{80\%, 60\%\}$. The SEP bound (RP-ML App.) expression in (13) of Theorem V.1 has an asymptotic performance close to the RP-detector $\mathbf{s}_{\text{RP-ML}}$. A similar trend is observed between the original dimension ML detector in (3) and its corresponding ML approximate.

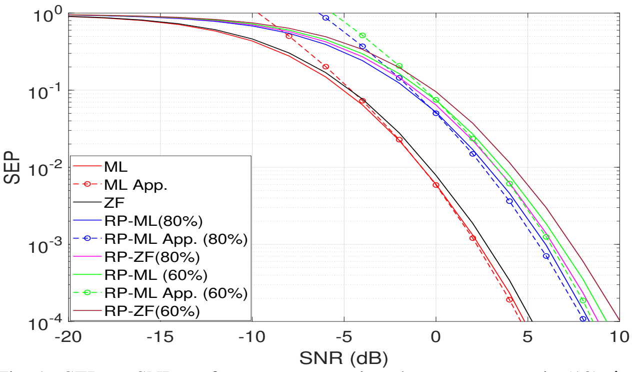

Fig. 1. SEP vs. SNR performance comparison between $\mathbf{s}_{\text{RP-ML}}$ in (10), $\hat{\mathbf{s}}_{\text{RP-ZF}}$ from (12), ML in (3), ZF in (5), the approximate SEP for ML and the RP-ML in Theorem V.1 for compression ratio $\frac{K}{N} \in \{80\%, 60\%\}$ and $\sigma_s^2 = 8$dB.

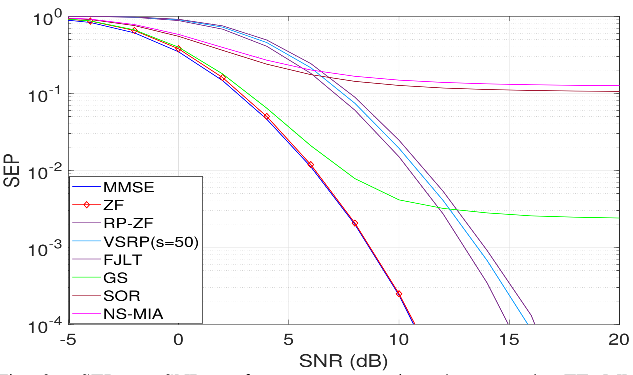

Fig. 2. SEP vs. SNR performance comparison between the ZF, MMSE [1], NS-MIA [10], Gauss-Seidel and SOR based detectors with the proposed $\hat{\mathbf{s}}_{\text{RP-ZF}}$, $\hat{\mathbf{s}}_{\text{VSRP}}$ and $\hat{\mathbf{s}}_{\text{FJLT}}$ in Section IV for a compression ratio of 70%.

Fig. 2 compares the SEP vs. SNR performance between the proposed $\hat{\mathbf{s}}_{\text{RP-ZF}}$, $\hat{\mathbf{s}}_{\text{VSRP}}$ and $\hat{\mathbf{s}}_{\text{FJLT}}$ for $\frac{K}{N} = 70\%$, and the benchmark detectors ZF, MMSE, along with the approximation based NS-MIA detector [10, (9)], and iterative Gauss-Seidel [4, (6)] and SOR $(\omega = 0.9)$ [6, (10)] based detectors, for $128 \times 30-$MIMO and $\sigma_s^2 = 8$dB. The sparsity parameter s in $\hat{\mathbf{s}}_{\text{VSRP}}$ is $\text{s} = 50$. The number of terms in the summation of NS-MIA based detector, the number of iterations of Gauss-Seidel, and SOR detectors are 10. It is observed that plots corresponding to $\hat{\mathbf{s}}_{\text{RP-ZF}}$, $\hat{\mathbf{s}}_{\text{VSRP}}$ and $\hat{\mathbf{s}}_{\text{FJLT}}$, *i.e.*, Rademacher, VSRP, and FJLT, are close to each other. It is worth noting that the performance gap between the optimal and the proposed RP-based detectors remain constant between moderate to low error probabilities. However, the SEP performance of the NS-MIA along with the Gauss-Seidel and SOR based detectors fail to improve with an increase in SNR and saturates.

VIII. Conclusion

This work proposed random projection based time complexity efficient detectors for the uplink massive MIMO wireless communication networks. The approximate SEP performance was derived for the proposed random projection-based fast detectors. The theoretical analysis and the simulation comparisons validated the trade-off between the detection performance and the time complexity of the proposed detectors with respect to the baseline detectors.

References

[1] S. Yang and L. Hanzo, "Fifty years of MIMO detection: The road to large-scale MIMOs," *IEEE communications surveys & tutorials*, vol. 17, no. 4, pp. 1941–1988, 2015.

[2] F. Rusek, D. Persson, B. K. Lau, E. G. Larsson, T. L. Marzetta, O. Edfors, and F. Tufvesson, "Scaling up MIMO: Opportunities and challenges with very large arrays," *IEEE signal processing magazine*, vol. 30, no. 1, pp. 40–60, 2012.

[3] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to algorithms*. MIT press, 2022.

[4] L. Dai, X. Gao, X. Su, S. Han, C.-L. I, and Z. Wang, "Low-complexity soft-output signal detection based on Gauss–Seidel method for uplink multiuser large-scale MIMO systems," *IEEE Transactions on Vehicular Technology*, vol. 64, no. 10, pp. 4839–4845, 2015.

[5] V. Gupta, A. K. Sah, and A. K. Chaturvedi, "Iterative matrix inversion based low complexity detection in large/massive MIMO systems," in *2016 IEEE International Conference on Communications Workshops (ICC)*, pp. 712–717, 2016.

[6] X. Gao, L. Dai, Y. Hu, Z. Wang, and Z. Wang, "Matrix inversion-less signal detection using SOR method for uplink large-scale MIMO systems," in *2014 IEEE Global Communications Conference*, pp. 3291–3295, 2014.

[7] L. Li and J. Hu, "Fast-converging and low-complexity linear massive MIMO detection with L-BFGS method," *IEEE Transactions on Vehicular Technology*, vol. 71, no. 10, pp. 10656–10665, 2022.

[8] M. A. Albreem, M. Juntti, and S. Shahabuddin, "Massive MIMO detection techniques: A survey," *IEEE Communications Surveys and Tutorials*, vol. 21, no. 4, pp. 3109–3132, 2019.

[9] H. Prabhu, J. Rodrigues, O. Edfors, and F. Rusek, "Approximative matrix inverse computations for very-large MIMO and applications to linear pre-coding systems," in *2013 IEEE Wireless Communications and Networking Conference (WCNC)*, pp. 2710–2715, IEEE, 2013.

[10] D. Zhu, B. Li, and P. Liang, "On the matrix inversion approximation based on neumann series in massive MIMO systems," in *2015 IEEE international conference on communications (ICC)*, pp. 1763–1769, IEEE, 2015.

[11] B. J. William and J. Lindenstrauss, "Extensions of lipschitz mapping into hilbert space," *Contemporary mathematics*, vol. 26, no. 189-206, pp. 323–341, 1984.

[12] A. Z. Broder, M. Charikar, A. M. Frieze, and M. Mitzenmacher, "Min-wise independent permutations," in *Proceedings of the thirtieth annual ACM symposium on Theory of computing*, pp. 327–336, 1998.

[13] M. S. Charikar, "Similarity estimation techniques from rounding algorithms," in *Proceedings of the thiry-fourth annual ACM symposium on Theory of computing*, pp. 380–388, 2002.

[14] P. Indyk, "Stable distributions, pseudorandom generators, embeddings, and data stream computation," *Journal of the ACM (JACM)*, vol. 53, no. 3, pp. 307–323, 2006.

[15] D. P. Woodruff *et al.*, "Sketching as a tool for numerical linear algebra," *Foundations and Trends® in Theoretical Computer Science*, vol. 10, no. 1–2, pp. 1–157, 2014.

[16] A. Blum, J. Hopcroft, and R. Kannan, *Foundations of data science*. Cambridge University Press, 2020.

[17] D. Achlioptas, "Database-friendly random projections: Johnson-lindenstrauss with binary coins," *Journal of computer and System Sciences*, vol. 66, no. 4, pp. 671–687, 2003.

[18] P. Li, T. J. Hastie, and K. W. Church, "Very sparse random projections," in *Proceedings of the 12th ACM SIGKDD international conference on Knowledge discovery and data mining*, pp. 287–296, 2006.

[19] N. Ailon and B. Chazelle, "Approximate nearest neighbors and the fast johnson-lindenstrauss transform," in *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*, pp. 557–563, 2006.

[20] S. Dasgupta and A. Gupta, "An elementary proof of a theorem of johnson and lindenstrauss," *Random Structures & Algorithms*, vol. 22, no. 1, pp. 60–65, 2003.

[21] J. G. Proakis and M. Salehi, "Digital communications, 5th expanded ed," 2007.

[22] S. M. Kay, *Fundamentals of statistical processing, Volume 2: Detection theory*. Pearson Education India, 2009.

[23] M. A. Albreem, W. Salah, A. Kumar, M. H. Alsharif, A. H. Rambe, M. Jusoh, and A. N. Uwaechia, "Low complexity linear detectors for massive MIMO: A comparative study," *IEEE Access*, vol. 9, pp. 45740–45753, 2021.