CS-671: Deep Learning and its Applications
Lecture: 01
Basics of Machine Learning: Linear Regression

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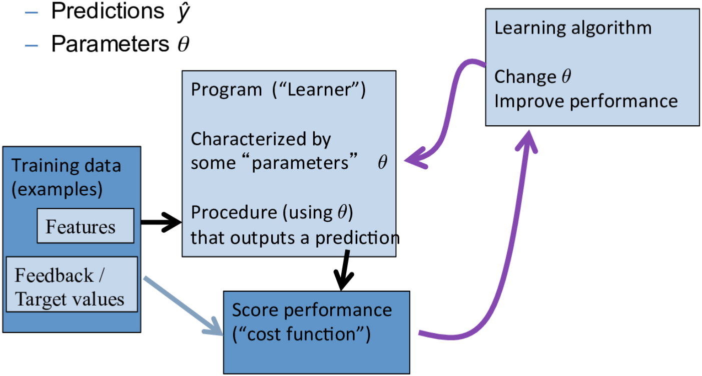
Presentation for CS-671@IIT Mandi (20 February, 2019)
(*Slides Credit : Intro to Machine Learning by Alexander Ihler)
https://www.youtube.com/watch?v=qPhMX0vb6D8&list=PLaXDtXvwY-oDvedS3f4HW0b4KxqpJ_imw

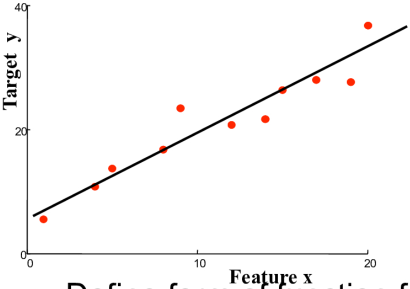
February - May, 2019

Supervised learning

- Notation
  - Features x
  - Targets y
  - Predictions $\hat{y}$
  - Parameters θ



Linear regression



"Predictor":
Evaluate line:

$$r = \theta_0 + \theta_1 x_1$$

return r

- Define form of function f(x) explicitly
- Find a good f(x) within that family

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Notation

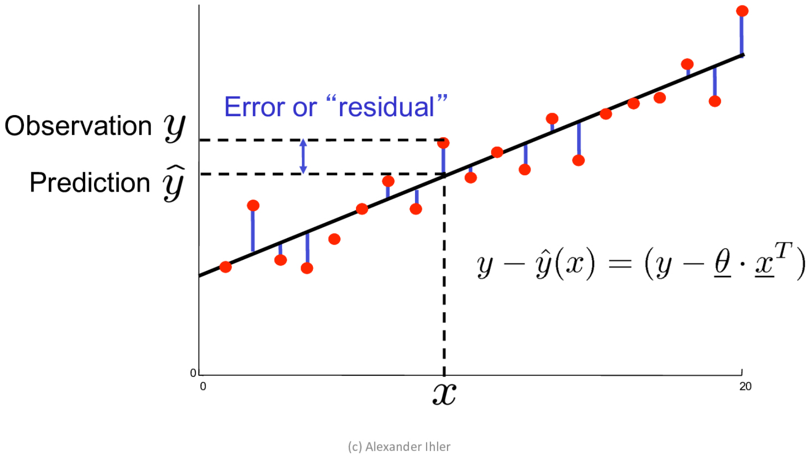
$$\hat{y}(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \dots$$

Define "feature" $x_0 = 1$ (constant)
Then

$$\hat{y}(x) = \theta\, x^T$$

$$\underline{\theta} = [\theta_0, \dots, \theta_n]$$
$$\underline{x} = [1, x_1, \dots, x_n]$$

Measuring error



Mean squared error

- How can we quantify the error?

$$\text{MSE}, \; J(\underline{\theta}) = \frac{1}{m}\sum_j (y^{(j)} - \hat{y}(x^{(j)}))^2$$
$$= \frac{1}{m}\sum_j (y^{(j)} - \underline{\theta} \cdot \underline{x}^{(j)T})^2$$

- Could choose something else, of course…
  - Computationally convenient (more later)
  - Measures the variance of the residuals
  - Corresponds to likelihood under Gaussian model of "noise"

$$\mathcal{N}(y \; ; \; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}$$

MSE cost function

$$\text{MSE},\ J(\underline{\theta}) = \frac{1}{m}\sum_j (y^{(j)} - \hat{y}(x^{(j)}))^2$$

$$= \frac{1}{m}\sum_j (y^{(j)} - \underline{\theta}\cdot\underline{x}^{(j)T})^2$$

- Rewrite using matrix form

$$\underline{\theta} = [\theta_0, \ldots, \theta_n]$$

$$\underline{y} = \left[y^{(1)} \ldots, y^{(m)}\right]^T$$

$$\underline{X} = \begin{bmatrix} x_0^{(1)} & \ldots & x_n^{(1)} \\ \vdots & \ddots & \vdots \\ x_0^{(m)} & \ldots & x_n^{(m)} \end{bmatrix}$$

$$J(\underline{\theta}) = \frac{1}{m}(\underline{y}^T - \underline{\theta}\,\underline{X}^T)\cdot(\underline{y}^T - \underline{\theta}\,\underline{X}^T)^T$$

```
# Python / NumPy:
e = Y – X.dot( theta.T );
J = e.T.dot( e ) / m  # = np.mean( e ** 2 )
```

Supervised learning

- Notation
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  - Targets y
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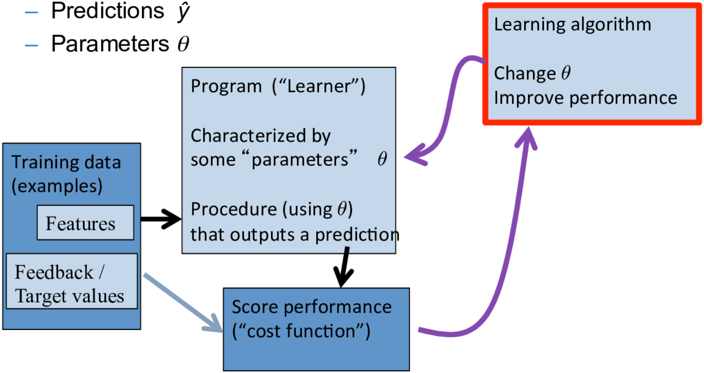
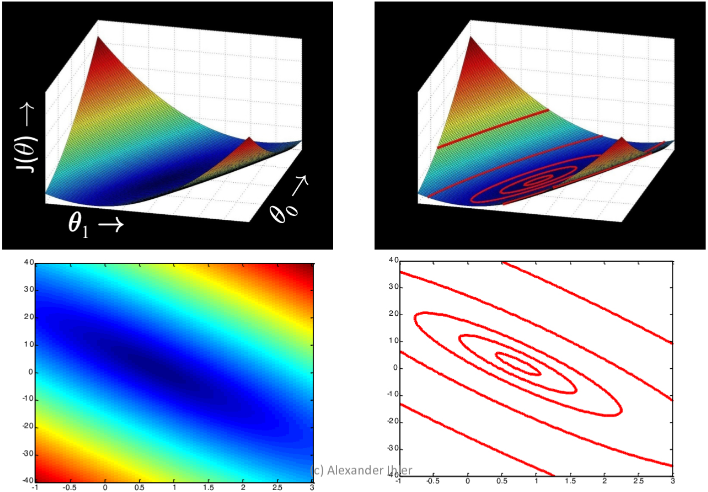
  - Parameters θ

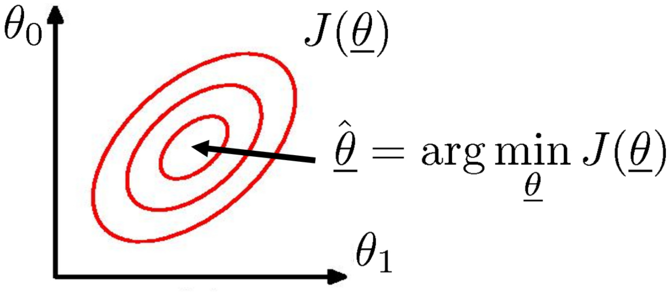


Visualizing the cost function



Finding good parameters

- Want to find parameters which minimize our error…
- Think of a cost "surface": error residual for that θ…



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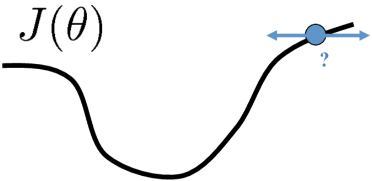
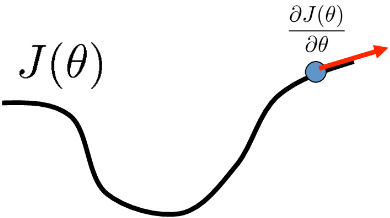
Linear regression: Gradient descent & stochastic gradient descent

Gradient descent



- How to change θ to improve J(θ)?
- Choose a direction in which J(θ) is decreasing

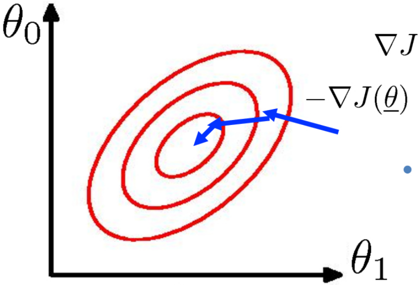
Gradient descent



- How to change θ to improve J(θ)?
- Choose a direction in which J(θ) is decreasing
- Derivative $\frac{\partial J(\theta)}{\partial \theta}$

- Positive => increasing
- Negative => decreasing

Gradient descent in more dimensions



- Gradient vector

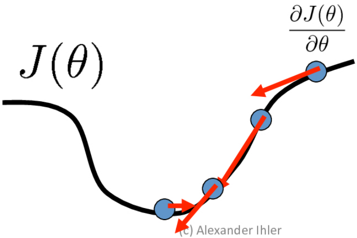
$$\nabla J(\underline{\theta}) = \left[\frac{\partial J(\underline{\theta})}{\partial \theta_0} \quad \frac{\partial J(\underline{\theta})}{\partial \theta_1} \quad \ldots \right]$$

- Indicates direction of steepest ascent
  (negative = steepest descent)

Gradient descent

- Initialization
- Step size
  - Can change as a function of iteration
- Gradient direction
- Stopping condition

```
Initialize θ
Do {
  θ ← θ - α ∇_θ J(θ)
} while ( α||∇J|| > ε )
```



Gradient for the MSE

- MSE $$J(\underline{\theta}) = \frac{1}{m}\sum_j (y^{(j)} - \underline{\theta}\cdot\underline{x}^{(j)T})^2$$

- ∇ J = ?

$$J(\underline{\theta}) = \frac{1}{m}\sum_j (\overbrace{y^{(j)} - \theta_0 \underline{x}_0^{(j)} - \theta_1 \underline{x}_1^{(j)} - \ldots}^{e_j(\theta)})^2$$

$$\frac{\partial J}{\partial \theta_0} = \frac{\partial}{\partial \theta_0}\frac{1}{m}\sum_j (\ e_j(\theta)\ )^2$$
$$= \frac{1}{m}\sum_j \frac{\partial}{\partial \theta_0}(\ e_j(\theta)\ )^2$$
$$= \frac{1}{m}\sum_j 2e_j(\theta)\ \frac{\partial}{\partial \theta_0} e_j(\theta)$$

$$\frac{\partial}{\partial \theta_0} e_j(\theta) = \frac{\partial}{\partial \theta_0} y^{(j)} - \frac{\partial}{\partial \theta_0}\theta_0 x_0^{(j)} - \frac{\partial}{\partial \theta_0}\theta_1 x_1^{(j)} - \ldots$$

0 0

$$= -x_0^{(j)}$$

Gradient for the MSE

- MSE $J(\underline{\theta}) = \frac{1}{m} \sum_j (y^{(j)} - \underline{\theta} \cdot \underline{x}^{(j)T})^2$

- ∇ J = ?

$$J(\underline{\theta}) = \frac{1}{m} \sum_j (\underbrace{y^{(j)} - \theta_0 \underline{x}_0^{(j)} - \theta_1 \underline{x}_1^{(j)} - \dots}_{e_j(\theta)})^2$$

$$\nabla J(\underline{\theta}) = \left[\quad \frac{\partial J}{\partial \theta_0} \qquad \frac{\partial J}{\partial \theta_1} \qquad \dots \right]$$

$$= \left[\frac{2}{m} \sum_j -e_j(\theta) x_0^{(j)} \quad \frac{2}{m} \sum_j -e_j(\theta) x_1^{(j)} \qquad \dots \right]$$

Gradient descent

- Initialization
- Step size
  - Can change as a function of iteration
- Gradient direction
- Stopping condition

Initialize θ

Do {

$\theta \leftarrow \theta - \alpha \nabla_\theta J(\theta)$

} while ($\alpha||\nabla J|| > \epsilon$)

$$J(\underline{\theta}) = \frac{1}{m}\sum_j (y^{(j)} - \underline{\theta} \cdot \underline{x}^{(j)T})^2$$

$$\nabla J(\underline{\theta}) = -\frac{2}{m}\sum_j \underbrace{(y^{(j)} - \underline{\theta} \cdot \underline{x}^{(j)T})}_{\text{Error magnitude \& direction for datum j}} \cdot \underbrace{[x_0^{(j)} x_1^{(j)} \ldots]}_{\text{Sensitivity to each } \theta_i}$$

Error magnitude & direction for datum j

Sensitivity to each θ_i

Derivative of MSE

$$\nabla J(\underline{\theta}) = -\frac{2}{m}\sum_j (y^{(j)} - \underline{\theta}\cdot\underline{x}^{(j)T}) \cdot [x_0^{(j)}\, x_1^{(j)} \ldots]$$

Error magnitude & direction for datum j

Sensitivity to each θ_i

- Rewrite using matrix form

$$\underline{\theta} = [\theta_0, \ldots, \theta_n]$$

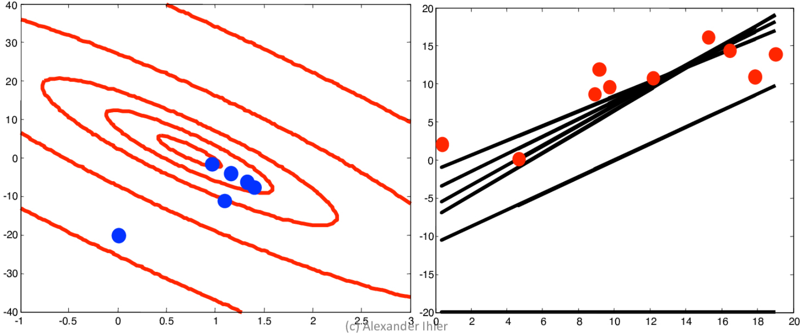
$$\underline{y} = \left[y^{(1)} \ldots, y^{(m)}\right]^T$$

$$\underline{X} = \begin{bmatrix} x_0^{(1)} & \ldots & x_n^{(1)} \\ \vdots & \ddots & \vdots \\ x_0^{(m)} & \ldots & x_n^{(m)} \end{bmatrix}$$

$$\nabla J(\underline{\theta}) = -\frac{2}{m}(\underline{y}^T - \underline{\theta X}^T) \cdot \underline{X}$$

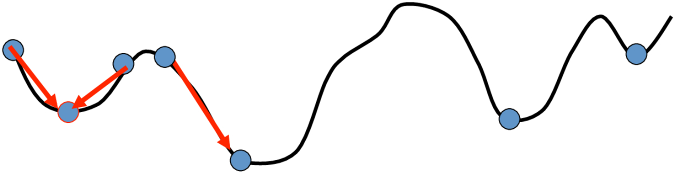
```
e = Y – X.dot( theta.T ); # error residual
DJ = - e.dot(X) * 2.0/m  # compute the gradient
theta -= alpha * DJ      # take a step
```

Gradient descent on cost function



Comments on gradient descent

- Very general algorithm
  - we' ll see it many times
- Local minima
  - Sensitive to starting point



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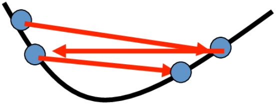
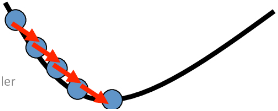
Comments on gradient descent

- Very general algorithm
  - we' ll see it many times
- Local minima
  - Sensitive to starting point
- Step size
  - Too large? Too small? Automatic ways to choose?
  - May want step size to decrease with iteration
  - Common choices:
    - Fixed
    - Linear: C/(iteration)
    - Line search / backoff (Armijo, etc.)
    - Newton' s method





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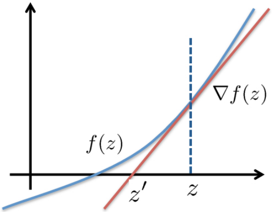
Newton's method



- Want to find the roots of f(x)
  - "Root": value of x for which f(x)=0
- Initialize to *some* point x
- Compute the tangent at x & compute where it crosses x-axis

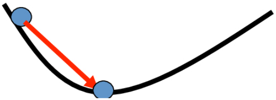
$$\nabla f(z) = \frac{0 - f(z)}{z' - z} \quad \Rightarrow \quad z' = z - \frac{f(z)}{\nabla f(z)}$$

- Optimization: find roots of ∇J(θ)

("Step size" λ = 1/∇∇J ; inverse curvature)

$$\nabla\nabla J(\theta) = \frac{0 - \nabla J(\theta)}{\theta' - \theta} \quad \Rightarrow \quad \theta' = \theta - \frac{\nabla J(\theta)}{\nabla\nabla J(\theta)}$$

  - Does not always converge; sometimes unstable
  - If converges, usually very fast
  - Works well for smooth, non-pathological functions, locally quadratic



(Multivariate:
 ∇ J(θ) = gradient vector
 ∇^2 J(θ) = matrix of 2^{nd} derivatives
 a/b = a b^{-1}, matrix inverse)

Stochastic / Online Gradient Descent

- MSE

$$J(\underline{\theta}) = \frac{1}{m}\sum_j J_j(\underline{\theta}), \qquad J_j(\underline{\theta}) = (y^{(j)} - \underline{\theta}\cdot\underline{x}^{(j)T})^2$$

- Gradient

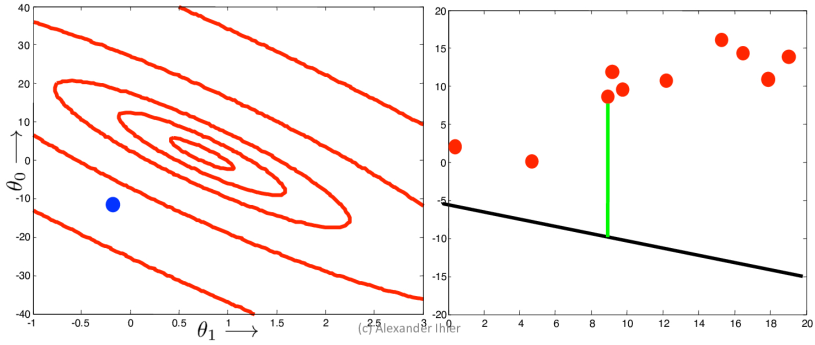
$$\nabla J(\underline{\theta}) = \frac{1}{m}\sum_j \nabla J_j(\underline{\theta}) \qquad \nabla J_j(\underline{\theta}) = (y^{(j)} - \underline{\theta}\cdot\underline{x}^{(j)T})\cdot[x_0^{(j)} x_1^{(j)} \ldots]$$

- Stochastic (or "online") gradient descent:
  - Use updates based on individual datum j, chosen at random
  - At optima, $\mathbb{E}\left[\nabla J_j(\underline{\theta})\right] = \nabla J(\underline{\theta}) = 0$
    (average over the data)

Online gradient descent

- Update based on each datum at a time
  - Find residual and the gradient of its part of the error & update

```
Initialize θ
Do {
 for j=1:m
   θ ← θ - α ∇_θ J_j(θ)
} while (not done)
```



Online gradient descent

- Update based on each datum at a time
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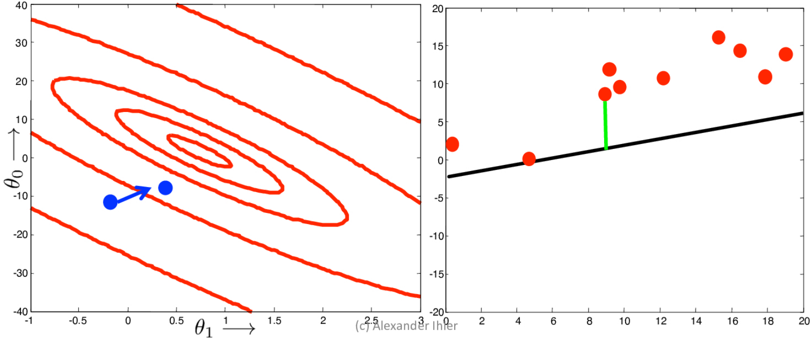
} while (not done)



Online gradient descent

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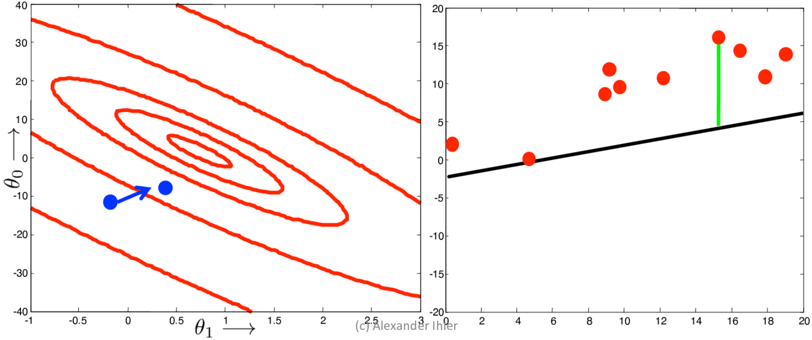

Online gradient descent

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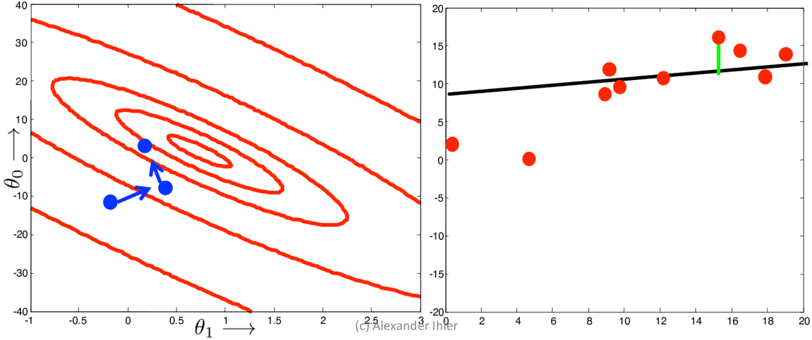

Online gradient descent

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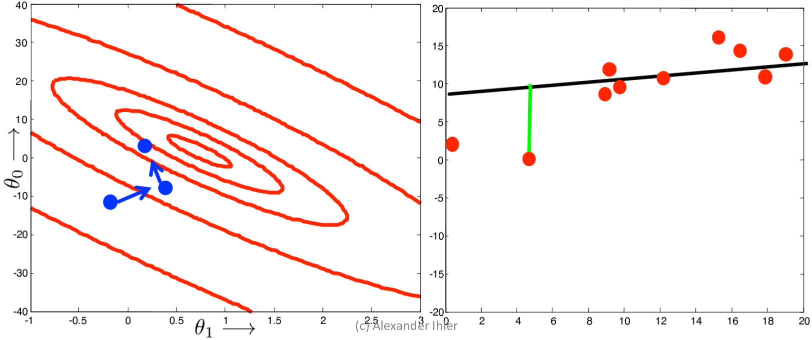

Online gradient descent

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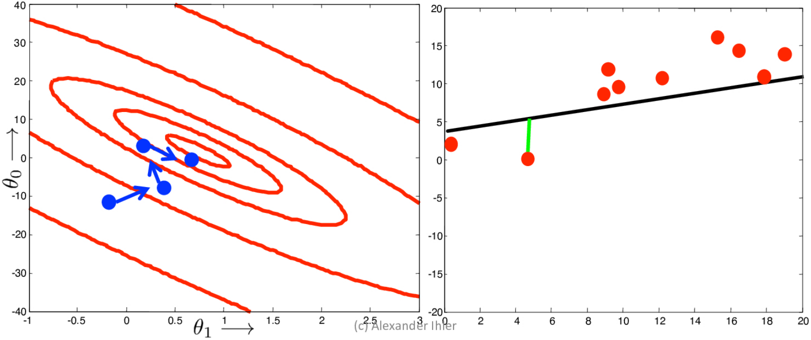


Online gradient descent

$$J_j(\underline{\theta}) = (y^{(j)} - \underline{\theta} \cdot \underline{x}^{(j)^T})^2$$

$$\nabla J_j(\underline{\theta}) = -2(y^{(j)} - \underline{\theta} \cdot \underline{x}^{(j)^T}) \cdot [x_0^{(j)} x_1^{(j)} \ldots]$$

```
Initialize θ
Do {
 for j=1:m
   θ ← θ - α ∇θ Jj(θ)
} while (not converged)
```

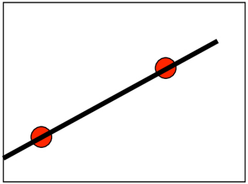
- Benefits
  - Lots of data = many more updates per pass
  - Computationally faster
- Drawbacks
  - No longer strictly "descent"
  - Stopping conditions may be harder to evaluate
    (Can use "running estimates" of J(.), etc.)
- Related: mini-batch updates, etc.

Linear regression: direct minimization

MSE Minimum

- Consider a simple problem
  - One feature, two data points
  - Two unknowns: θ_0, θ_1
  - Two equations:

$$y^{(1)} = \theta_0 + \theta_1 x^{(1)}$$
$$y^{(2)} = \theta_0 + \theta_1 x^{(2)}$$



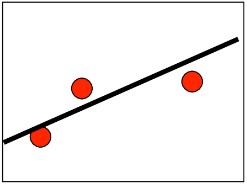
- Can solve this system directly:

$$\underline{y}^T = \underline{\theta}\, \underline{X}^T \qquad \Rightarrow \qquad \underline{\hat{\theta}} = y^T (\underline{X}^T)^{-1}$$

- However, most of the time, m > n
  - There may be no linear function that hits all the data exactly
  - Instead, solve directly for minimum of MSE function

SSE Minimum

$$\nabla J(\underline{\theta}) = -(\underline{y}^T - \underline{\theta}\underline{X}^T) \cdot \underline{X} \quad = \quad \underline{0}$$



- Reordering, we have

$$\underline{y}^T\underline{X} - \underline{\theta}\underline{X}^T \cdot \underline{X} \quad = \quad \underline{0}$$
$$\underline{y}^T\underline{X} = \underline{\theta}\underline{X}^T \cdot \underline{X}$$
$$\underline{\theta} \quad = \quad \underline{y}^T\underline{X}(\underline{X}^T\underline{X})^{-1}$$

- $X (X^T X)^{-1}$ is called the "pseudo-inverse"

- If X^T is square and independent, this is the inverse
- If m > n: overdetermined; gives minimum MSE fit

Python SSE

- This is easy to solve in Python / NumPy…

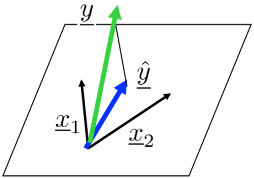
$$\underline{\theta} = \underline{y}^T \underline{X} (\underline{X}^T \underline{X})^{-1}$$

```
#   y = np.matrix( [[y1], … , [ym]] )
#   X = np.matrix( [[x1_0 … x1_n], [x2_0 … x2_n], …] )

% Solution 1: "manual"
   th = y.T * X * np.linalg.inv(X.T * X);

% Solution 2: "least squares solve"
   th = np.linalg.lstsq(X, Y);
```

Normal equations

$$\nabla J(\underline{\theta}) = 0 \quad \Rightarrow \quad (\underline{y}^T - \underline{\theta X}^T) \cdot \underline{X} \quad = \quad \underline{0}$$

- Interpretation:
  - (y - θX) = (y - yhat) is the vector of errors in each example
  - X are the features we have to work with for each example
  - Dot product = 0: orthogonal

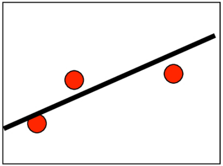


$$\underline{y}^T = [y^{(1)} \dots y^{(m)}]$$
$$\underline{x}_i = [x_i^{(1)} \dots x_i^{(m)}]$$

Normal equations

$$\nabla J(\underline{\theta}) = 0 \quad \Rightarrow \quad (\underline{y}^T - \underline{\theta X}^T) \cdot \underline{X} \quad = \quad \underline{0}$$

- Interpretation:
  - (y - θX) = (y - yhat) is the vector of errors in each example
  - X are the features we have to work with for each example
  - Dot product = 0: orthogonal
- Example:

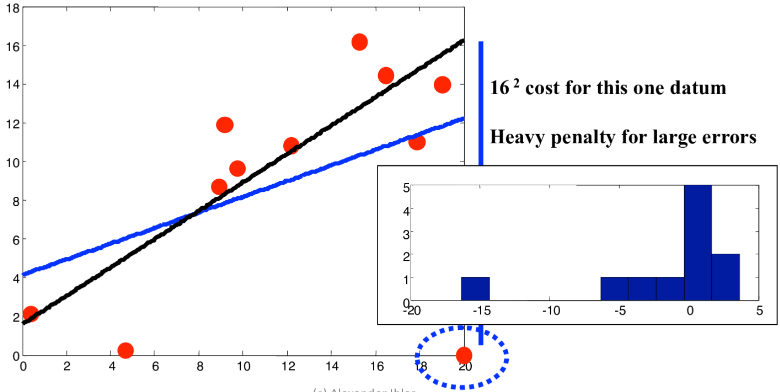


$$\underline{y} = [1 \;\; 3 \;\; 3]^T$$
$$\underline{x}_0 = [1 \;\; 1 \;\; 1]^T \qquad \theta = [1.00 \;\; 0.57]$$
$$\underline{x}_1 = [1 \;\; 2 \;\; 4]^T$$

$$\underline{e} = (y - \hat{y}) = [-0.57 \;\; 0.85 \quad -0.28]^T$$

Effects of MSE choice

- Sensitivity to outliers



16^2 cost for this one datum

Heavy penalty for large errors

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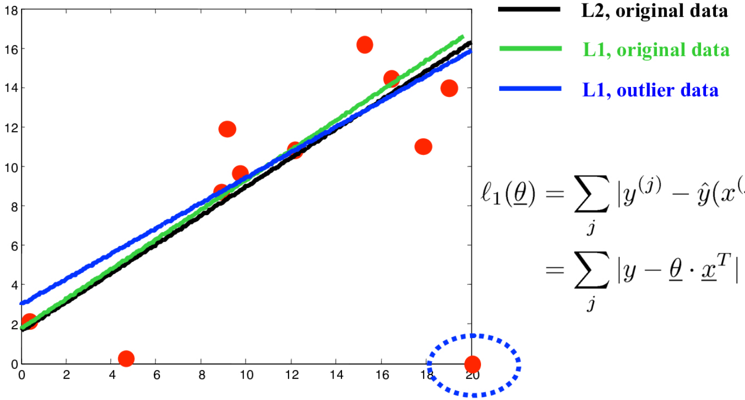
L1 error



$$\ell_1(\underline{\theta}) = \sum_j |y^{(j)} - \hat{y}(x^{(j)})|$$
$$= \sum_j |y - \underline{\theta} \cdot \underline{x}^T|$$

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Cost functions for regression

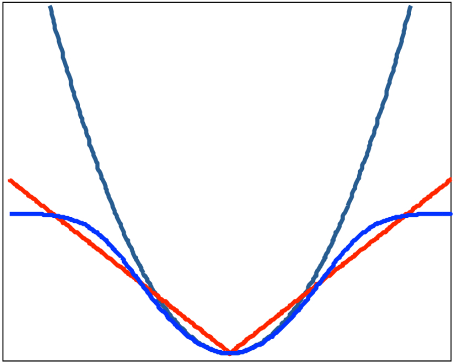
$\ell_2 \ : \ (y - \hat{y})^2$ **(MSE)**

$\ell_1 \ : \ |y - \hat{y}|$ **(MAE)**

Something else entirely…

$c - \log(\exp(-(y - \hat{y})^2) + c)$

(???)



$\leftarrow (y - \hat{y}) \rightarrow$

"Arbitrary" functions can't be solved in closed form…

- use gradient descent

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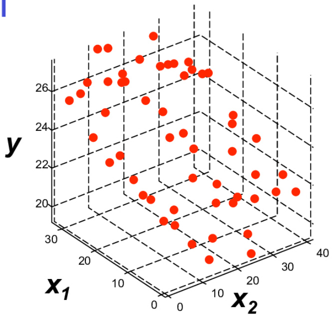
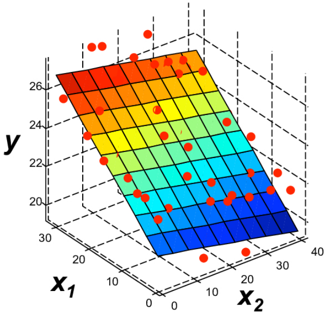
Linear regression: nonlinear features

More dimensions?



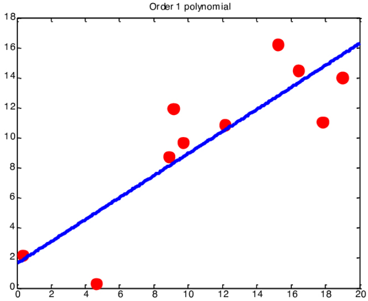


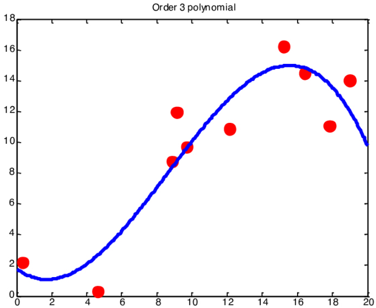
$$\hat{y}(x) = \underline{\theta} \cdot \underline{x}^T$$

$$\underline{\theta} = [\theta_0 \;\; \theta_1 \;\; \theta_2]$$
$$\underline{x} = [1 \;\; x_1 \;\; x_2]$$

Nonlinear functions

- What if our hypotheses are not lines?
  - Ex: higher-order polynomials





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Nonlinear functions

- Single feature x, predict target y:

$$D = \{(x^{(j)}, y^{(j)})\}$$

$$\hat{y}(x) = \theta_0 + \theta_1\, x + \theta_2\, x^2 + \theta_3\, x^3$$

$\Downarrow$ Add features:

$$D = \{([x^{(j)}, (x^{(j)})^2, (x^{(j)})^3], y^{(j)})\}$$

$\Downarrow$

$$\hat{y}(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3$$

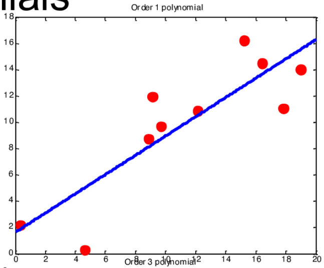
Linear regression in new features

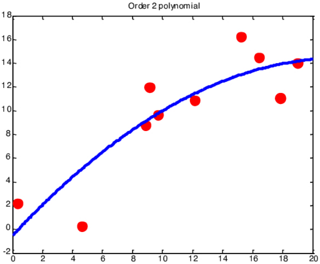
- Sometimes useful to think of "feature transform"

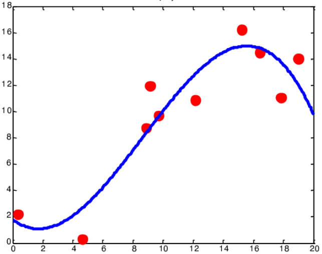
$$\Phi(x) = \left[1\,,\, x\,,\, x^2\,,\, x^3\,,\, \ldots\right] \qquad \hat{y}(x) = \underline{\theta} \cdot \Phi(x)$$

Higher-order polynomials

- Fit in the same way
- More "features"







Features

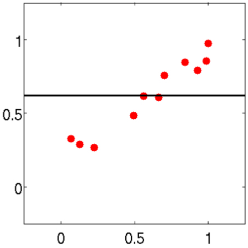
- In general, can use any features we think are useful

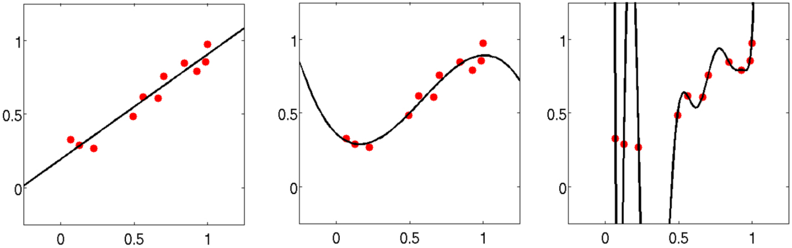
- Other information about the problem
  - Sq. footage, location, age, …
- Polynomial functions
  - Features [1, x, x^2, x^3, …]
- Other functions
  - 1/x, sqrt(x), x_1 * x_2, …

- "Linear regression" = linear in the parameters
  - Features we can make as complex as we want!

Higher-order polynomials

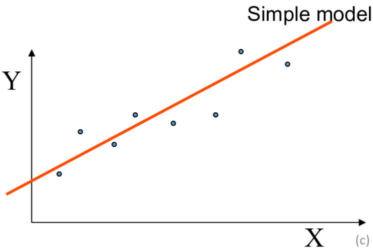
- Are more features better?
- "Nested" hypotheses
  - 2^{nd} order more general than 1^{st},
  - 3^{rd} order " " than 2^{nd}, …
- Fits the observed data better

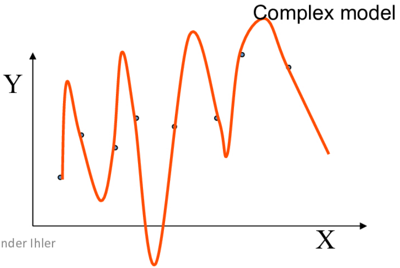




Overfitting and complexity

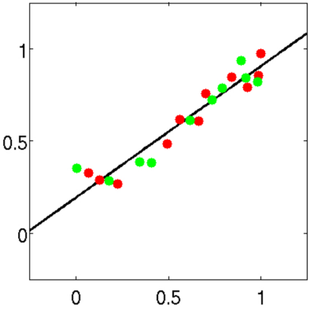
- More complex models will always fit the training data better
- But they may "overfit" the training data, learning complex relationships that are not really present





Test data

- After training the model
- Go out and get more data from the world
  - New observations (x,y)
- How well does our model perform?



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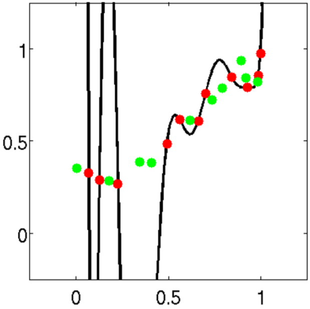
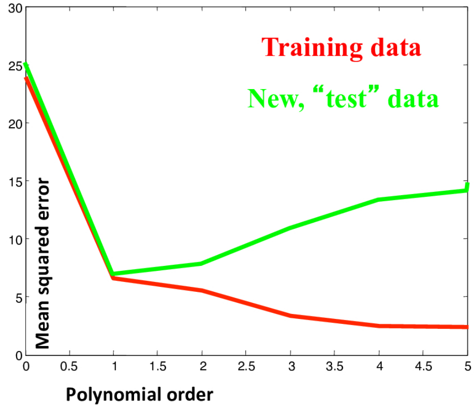


Training versus test error

- Plot MSE as a function of model complexity
  - Polynomial order
- Decreases
  - More complex function fits training data better
- What about new data?
- 0^{th} to 1^{st} order
  - Error decreases
  - Underfitting
- Higher order
  - Error increases
  - Overfitting



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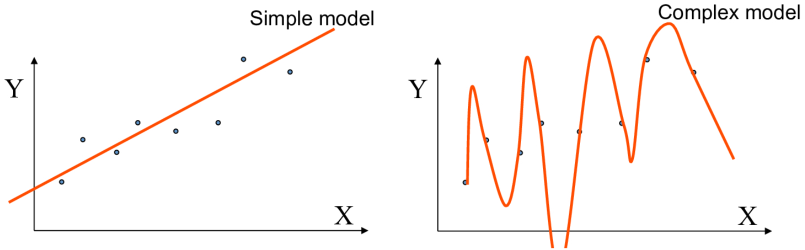
Linear regression: bias and variance

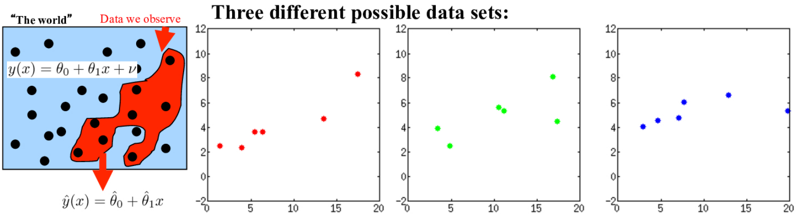
Inductive bias

- The assumptions needed to predict examples we haven't seen
- Makes us "prefer" one model over another
- Polynomial functions; smooth functions; etc

- Some bias is necessary for learning!



Bias & variance



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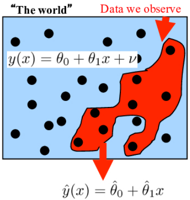
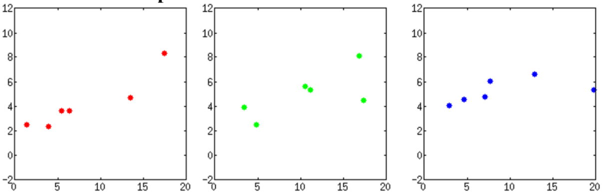
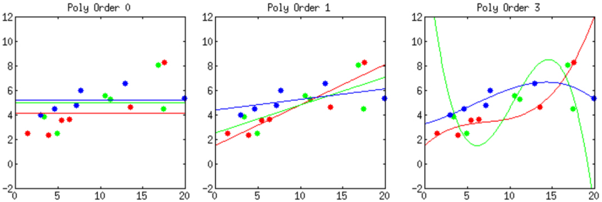
Bias & variance



Three different possible data sets:



Each would give different predictors for any polynomial degree:



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Detecting overfitting

- Overfitting effect
  - Do better on training data than on future data
  - Need to choose the “right” complexity

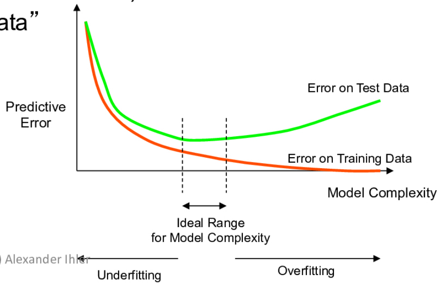
- One solution: “Hold-out” data
- Separate our data into two sets
  - Training
  - Test
- Learn only on training data
- Use test data to estimate generalization quality
  - Model selection

- All good competitions use this formulation
  - Often multiple splits: one by judges, then another by you

What to do about under/overfitting?

- Ways to increase complexity?
  - Add features, parameters
  - We'll see more…

- Ways to decrease complexity?
  - Remove features ("feature selection")
  - "Fail to fully memorize data"
    - Partial training
    - Regularization



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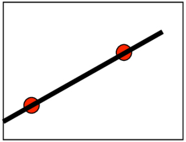
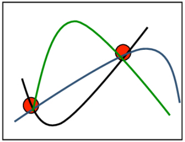
Linear regression: regularization

Linear regression

- Linear model, two data



- Quadratic model, two data?
  - Infinitely many settings with zero error
  - How to choose among them?



- Higher order coefficents = 0?
  - Uses knowledge of where features came from…
- Could choose e.g. minimum magnitude:

$$\min \underline{\theta\theta}^T \quad s.t. \quad J(\underline{\theta}) = 0$$

- A type of *bias*: tells us which models to prefer

Regularization

- Can modify our cost function J to add "preference" for certain parameter values

$$J(\underline{\theta}) = \frac{1}{2}(\underline{y} - \underline{\theta}\,\underline{X}^T) \cdot (\underline{y} - \underline{\theta}\,\underline{X}^T)^T + \alpha\,\theta\theta^T$$

L_2 penalty:
"Ridge regression"

- New solution (derive the same way)

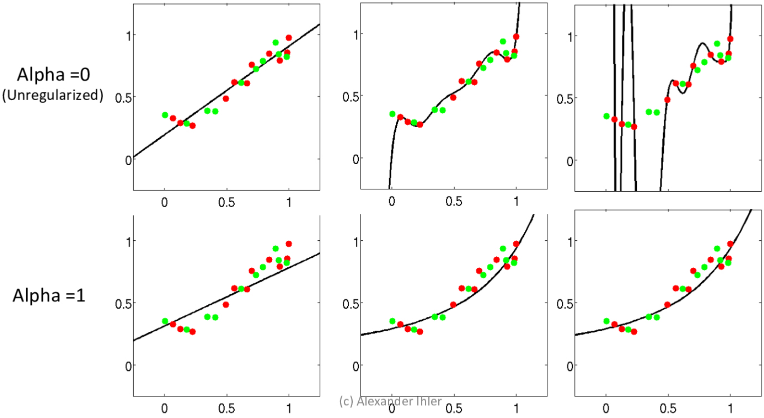
$$\underline{\theta} \quad = \quad \underline{y}\,\underline{X}(\underline{X}^T\,\underline{X} + \alpha I)^{-1}$$

  - Problem is now well-posed for any degree

- Notes:
  - "Shrinks" the parameters toward zero
  - Alpha large: we prefer small theta to small MSE
  - Regularization term is independent of the data: paying more attention reduces our model variance

Regularization

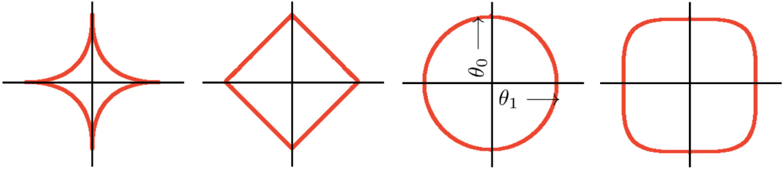
- Compare between unreg. & reg. results



Different regularization functions

- More generally, for the L_p regularizer: $\left(\sum_i |\theta_i|^p\right)^{\frac{1}{p}}$

Isosurfaces: $||\theta||_p$ = constant



p=0.5 | p=1 Lasso | p=2 Quadratic | p=4

L_0 = limit as $p \to 0$: "number of nonzero weights", a natural notion of complexity

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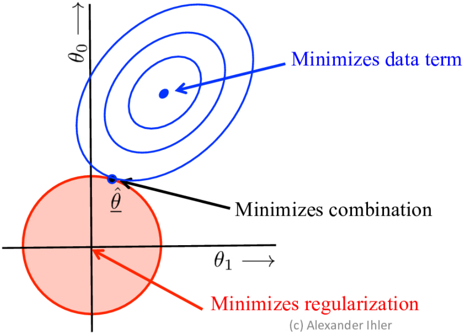
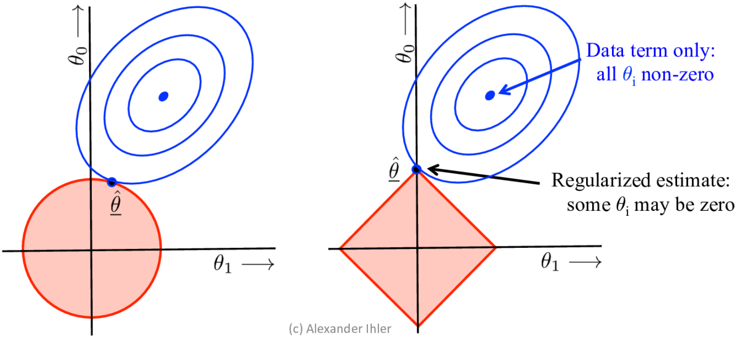
Regularization: L1 vs L2

- Estimate balances data term & regularization term



Regularization: L1 vs L2

- Estimate balances data term & regularization term
- Lasso tends to generate sparser solutions than a quadratic regularizer.

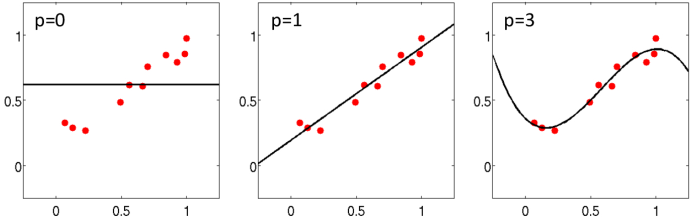


Linear regression: hold-out, cross-validation

Model selection

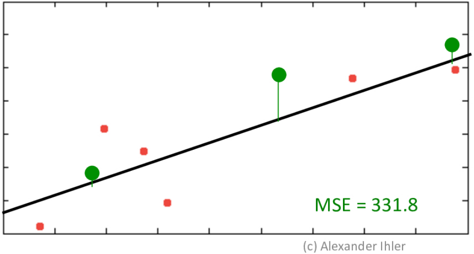
- Which of these models fits the data best?
  - p=0 (constant); p=1 (linear); p=3 (cubic); …
- Or, should we use KNN? Other methods?
- Model selection problem
  - Can't use training data to decide (esp. if models are nested!)
- Want to estimate $\mathbb{E}_{(x,y)}[J(y, \hat{y}(x\,;\,D))]$

J = loss function (MSE)
D = training data set



Hold-out method

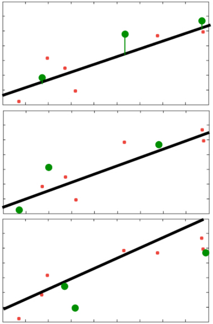
- Validation data
  - "Hold out" some data for evaluation (e.g., 70/30 split)
  - Train only on the remainder
- Some problems, if we have few data:
  - Few data in hold-out: noisy estimate of the error
  - More hold-out data leaves less for training!



	$x^{(i)}$	$y^{(i)}$
Training data	88	79
	32	-2
	27	30
	68	73
	7	-16
	20	43
Validation data	53	77
	17	16
	87	94

Cross-validation method

- K-fold cross-validation
  - Divide data into K disjoint sets
  - Hold out one set (= M / K data) for evaluation
  - Train on the others (= M*(K-1) / K data)



Split 1:
MSE = 331.8

Split 2:
MSE = 361.2

Split 3:
MSE = 669.8

3-Fold X-Val MSE
= 464.1

Training data

Validation data

$x^{(i)}$	$y^{(i)}$
88	79
32	-2
27	30
68	73
7	-16
20	43
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87	94

Cross-validation method

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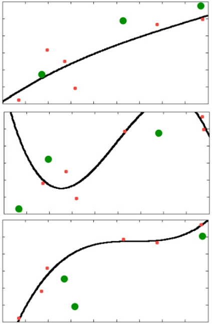


Split 1:
MSE = 280.5

Split 2:
MSE = 3081.3

Split 3:
MSE = 1640.1

3-Fold X-Val MSE
= 1667.3

Training data

Validation data

$x^{(i)}$	$y^{(i)}$
88	79
32	-2
27	30
68	73
7	-16
20	43
53	77
17	16
87	94

Cross-validation

- Advantages:
  - Lets us use more (M) validation data
    (= less noisy estimate of test performance)

- Disadvantages:
  - More work
    - Trains K models instead of just one
  - Doesn't evaluate any *particular* predictor
    - Evaluates K different models & averages
    - Scores *hyperparameters / procedure*, not an actual, specific predictor!

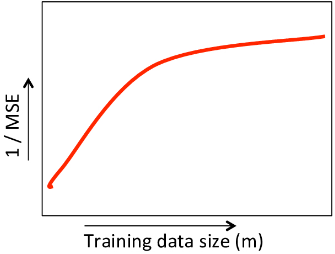
- Also: still estimating error for M' < M data…

Learning curves

- Plot performance as a function of training size
  - Assess impact of fewer data on performance

    Ex: MSE0 - MSE (regression)

    or 1-Err (classification)

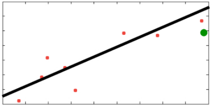


- Few data
  - More data significantly improve performance
- "Enough" data
  - Performance saturates

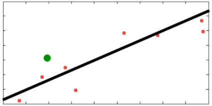
- If slope is high, decreasing *m* (for validation / cross-validation) might have a big impact…

Leave-one-out cross-validation

- When K=M (# of data), we get
  - Train on all data except one
  - Evaluate on the left-out data
  - Repeat M times (each data point held out once) and average



MSE = ...



MSE = ...

⋮

LOO X-Val MSE = ...

Training data

Validation data

$x^{(i)}$	$y^{(i)}$
88	79
32	-2
27	30
68	73
7	-16
20	43
53	77
17	16
87	94

Cross-validation Issues

- Need to balance:
  - Computational burden (multiple trainings)
  - Accuracy of estimated performance / error
- Single hold-out set:
  - Estimates performance with M' < M data (important? learning curve?)
  - Need enough data to trust performance estimate
  - Estimates performance of a particular, trained learner
- K-fold XVal
  - K times as much work, computationally
  - Better estimates, still of performance with M' < M data
- LOO XVal
  - M times as much work, computationally
  - M' $\approx$ M, but overall error estimate may have high variance