Stability and Hopf bifurcation analysis of the Mackey-Glass and Lasota equations

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Outline

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  - Physiological processes as control problems
- Models
  - Mackey-Glass equation
  - Lasota equation
- Style of analysis
  - Local stability
  - Local Hopf bifurcation
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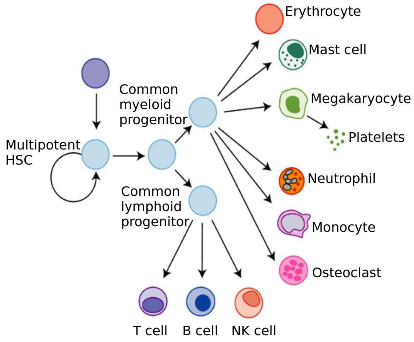
  - Trade offs/design considerations
- Another application: population dynamics
  - Some Models
  - Results

Motivation



Normal haematopoiesis
Expert reviews in Molecular Medicine © Cambridge University Press

- Haematopoiesis: Formation of blood cellular components
- Erythropoiesis: Formation of erythrocytes (red blood cells)

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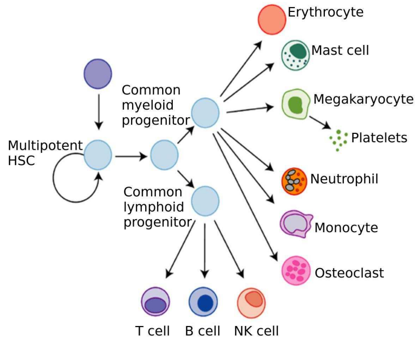

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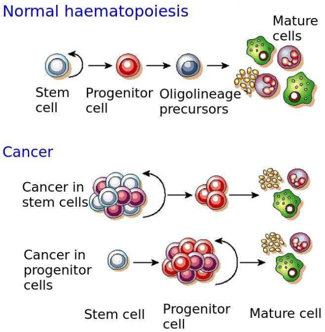

Motivation

"Attempt to devise novel therapies for disease by manipulating control parameters back into normal range"

"Demonstrate the onset of abnormal dynamics in animal bodies by gradual tuning of control parameters"

M.C. Mackey and L. Glass
Oscillation and chaos in physiological control systems
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Stability analysis

Stability conditions $\Rightarrow$ manipulate control parameters $\Rightarrow$ devise therapies

Bifurcation analysis

Alerts us about the onset of abnormal dynamics as parameters vary

Model

Mackey-Glass equation (haematopoiesis)

$$\dot{x}(t) = \beta \underbrace{\frac{x(t-\tau)}{1+x^n(t-\tau)}}_{F\left(x(t-\tau)\right)} - \gamma x(t)$$

$x(t) > 0$: number of blood cells

$\beta > 0$: dependence on the number of mature cells

$\tau > 0$: delay between cell production and release into blood stream

$\gamma > 0$: cell destruction rate

$n > 0$: captures non-linearity

Model

Lasota equation (erythropoiesis)

$$\dot{x}(t) = \beta \underbrace{x^n(t-\tau)e^{-x(t-\tau)}}_{F\left(x(t-\tau)\right)} - \gamma x(t)$$

$x(t) > 0$: number of red blood cells

$\beta > 0$: demand for oxygen

$\tau > 0$: time required for erythrocytes to attain maturity

$\gamma > 0$: cell destruction rate

$n > 0$: captures non-linearity

Local Stability

General form

$$\dot{x}(t) = \beta F\big(x(t-\tau)\big) - \gamma x(t)$$

Non-trivial equilibrium: x^*

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Sufficient condition

$$-\frac{x^* F'(x^*)}{F(x^*)}\gamma\tau < \frac{\pi}{2} \quad \Rightarrow \quad -F'(x^*)\beta\tau < \frac{\pi}{2}$$

Insights to ensure stability

$x^* F'(x^*)/F(x^*)$ and $\gamma\tau$ need to be bounded

Local Stability

Necessary and Sufficient condition

$$\tau\sqrt{\left(\beta F'(x^*)\right)^2-\gamma^2} < \cos^{-1}\left(\frac{\gamma}{\beta F'(x^*)}\right)$$

Hopf condition

$$\tau = \frac{\cos^{-1}\left(\frac{\gamma}{\beta F'(x^*)}\right)}{\sqrt{\left(\beta F'(x^*)\right)^2-\gamma^2}}$$

Period: $2\pi\tau/\cos^{-1}\left(\gamma/\beta F'(x^*)\right)$

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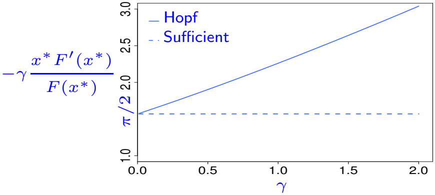

Figure: Stability chart ($\tau = 1$)
Region below the lines is stable

Hopf bifurcation $\Rightarrow$ limit cycles
Oscillations in cell count (limit cycles) $\Rightarrow$ pathological behaviour

Are the limit cycles orbitally stable ?

Local Hopf Bifurcation

Bifurcation parameter

- System parameters
  - may vary with time
  - affect the equilibrium
- Introduce non-dimensional parameter

$$\dot{x}(t) = \eta\Big(\beta F\big(x(t-\tau)\big) - \gamma x(t)\Big) = f_\eta(x)$$

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Transversality condition

At the Hopf condition, $\eta = \eta_c = 1$

$$\text{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\eta}\right)_{\eta=\eta_c} \neq 0$$

Local Hopf Bifurcation

Using Poincaré normal forms and center manifold theory[1]

- supercritical/subcritical Hopf
- orbital stability of limit cycles

[1] B. Hassard, N. Kazarinoff and Y. Wan, Theory and Applications of Hopf Bifurcation, 1981

Local Hopf Bifurcation

Using Poincaré normal forms and center manifold theory[1]

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Poincaré normal form

Two-dimensional, one parameter, system

$$\dot{x}_1 = \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2)$$

Introducing a *complex variable* $z = x_1 + ix_2$,
the Poincaré normal form can be written as

$$\dot{z} = (\alpha + i)z \pm \bar{z}z^2$$

[1] B. Hassard, N. Kazarinoff and Y. Wan, Theory and Applications of Hopf Bifurcation, 1981

Local Hopf Bifurcation

Center manifold theorem

Two dimensional system

$$x' = f(x, y), \qquad y' = g(x, y)$$

- Invariant manifold: $y = h(x)$ for small $|x|$ if solution with $x(0) = x_0$, $y(0) = h(x_0)$ lies on $y = h(x)$
- The local behaviour of the system can be analysed by studying the dynamics on the manifold
- Dimension of the system is hence reduced

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Example

$$x' = ax^3 + xy - xy^2, \qquad y' = -y + bx^2 + x^2 y$$

- *Invariant manifold*: $y = h(x) = bx^2 + O(x^4)$
- *Reduced system*

$$u' = au^3 + uh(u) - uh^2(u)$$
$$u' = (a + b)u^3 + O(u^5)$$

Local Hopf Bifurcation

Style of analysis

- Let $\dot{x} = f_\eta(x)$ be the non-linear system and q be the complex eigenvector of Jacobian $Df_\eta(x^*)$
- Reduce flow of $f_\eta(x)$ to a 2-manifold (center manifold) which is invariant under the flow that is tangential to the q-plane
- Rewrite dynamics on center manifold using a single complex variable
- Determine the sign of first lyapunov coefficient and floquet exponent to establish the type of Hopf and the orbital stability of limit cycles

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First lyapunov coefficient $\Rightarrow$ type of Hopf

$\mu_2 > 0$: supercritical $\quad$ $\mu_2 < 0$: subcritical

Floquet exponent $\Rightarrow$ stability of limit cycles

$\beta_2 > 0$: unstable $\quad$ $\beta_2 < 0$: stable

Numerical Example: Mackey-Glass Equation

- Parameter values: $\beta = 0.8, \gamma = 0.3, n = 10$
- *Hopf condition*: critical time delay $\tau_c = 1.14$, $\eta = 1$
- For $\eta = 1.05$

$$\mu_2 = 29.10 > 0 \qquad \beta_2 = -35.64 < 0$$

Hopf bifurcation is supercritical, and leads to orbitally stable limit cycles

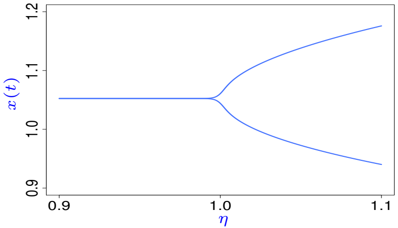


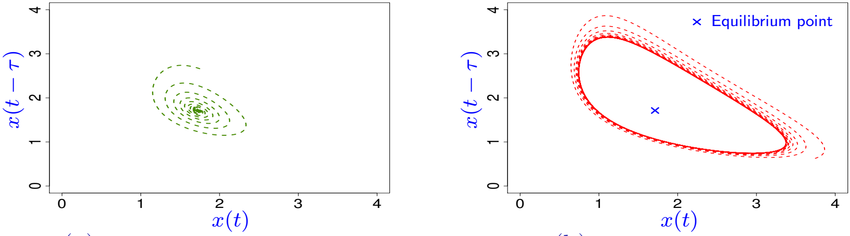
Figure: Bifurcation diagram. As η varies, limit cycles emerge

Numerical Example: Lasota Equation

- Parameter values: $\beta = 0.9, \gamma = 0.1, n = 0.1$
- *Hopf condition*: critical time delay $\tau_c = 17.69$, $\eta = 1$
- For $\eta = 1.05$

$$\mu_2 = 0.8072 > 0 \qquad \beta_2 = -0.0398 < 0$$

Orbitally stable limit cycles emerge from a supercritical Hopf



(a) Stable equilibrium ($\tau < \tau_c$)

(b) Limit cycle ($\tau > \tau_c$)

Figure: Phase portraits

Contributions

Stability analysis

- Sufficient condition
  - simple condition to guide models into normal range
  - can be helpful for design considerations to devise therapies
- Necessary and Sufficient condition
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Bifurcation analysis

Used a non-dimensional bifurcation parameter

- Emergence of limit cycles $\Rightarrow$ onset of abnormal behaviour
- Pathological behaviour could be predicted and prevented

Population Dynamics

Logistic equation

$$\dot{x}(t) = rx(t)\Big(1 - \big(x(t-\tau)/K\big)\Big)$$

delayed response to diminishing resources

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Logistic equation + constant harvesting rate [1]

$$\dot{x}(t) = rx(t)\Big(1 - \big(x(t-\tau)/K\big)\Big) - \gamma x(t)$$

[1] S.A.H. Geritz and É. Kisdi, Mathematical ecology: why mechanistic models?
Journal of Mathematical Biology, 2012

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- Necessary and Sufficient condition
- Hopf condition

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Population Dynamics

Perez-Malta-Coutinho equation

$$\dot{x}(t) = r\, \underbrace{x(t-\tau)\Big(1 - \big(x(t-\tau)/K\big)\Big)}_{F\left(x(t-\tau)\right)} - \gamma x(t)$$

shown to fit experimental data

Population Dynamics

Perez-Malta-Coutinho equation

$$\dot{x}(t) = r \underbrace{x(t-\tau)\Big(1 - \big(x(t-\tau)/K\big)\Big)}_{F\left(x(t-\tau)\right)} - \gamma x(t)$$

shown to fit experimental data

- Stability and Hopf bifurcation analysis
- Also exhibits chaos

Predator-Prey Dynamics

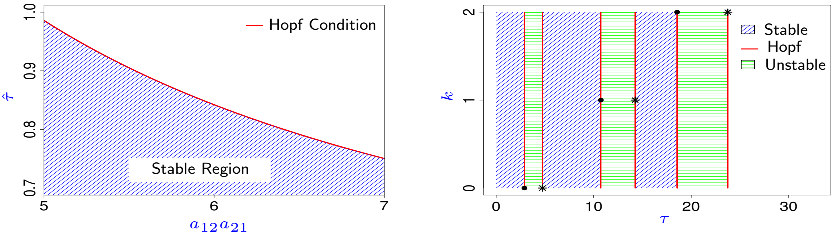
Lotka-Volterra logistic model with discrete delays

$$\dot{x}(t) = x(t)\big(r_1 - a_{11}x(t-\tau_1) - a_{12}y(t-\tau_2)\big)$$
$$\dot{y}(t) = y(t)\big(-r_2 + a_{21}x(t-\tau_3) - a_{22}y(t-\tau_4)\big)$$

$x(t)$	:	prey population
$y(t)$	:	predator population
$r_i, a_{ij} > 0$	:	model parameters, $i, j \in \{1, 2\}$
$\tau_i \geq 0$	:	time delays, $i \in \{1, 2, 3, 4\}$

S. Manjunath and G. Raina, A Lotka-Volterra time delayed system: stability switches and Hopf bifurcation analysis, *in Proceedings of* 26th Chinese Control and Decision Conference, 2014

Predator-Prey Dynamics



(a) Case 1 (b) Case 2

Figure: In Case 1, system becomes unstable as τ varies. In Case 2, system undergoes multiple stability switches as τ varies.

Some Background References

Physiology

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Population dynamics

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[2] J.F. Perez, C.P. Malta and F.A.B. Coutinho, Qualitative analysis of oscillations in isolated populations of flies, Journal of Theoretical Biology, 1978

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[4] J. Wang, X. Zhou and L. Huang, Hopf bifurcation and multiple periodic solutions in Lotka-Volterra systems with symmetries, Nonlinear Analysis: Real World Applications, 2013