When ^{133}Cs condensate meets ^{87}Rb condensate at finite temperature

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Plan of the talk

Bose-Einstein Condensation

- Macroscopic occupation of non-interacting bosons in the ground state/lowest single particle level of the system
- Gas of bosonic particles cooled below a critical temperature T_c condenses into an ideal Bose-Einstein condensate(BEC)
- Criteria for condensation @

$$\varpi = n\left(\frac{2\pi\hbar^2}{mKT}\right)^{3/2} \sim 1,$$

- De Broglie wavelength λ_{dB} comparable to the distance between the particles–wave packets start to overlap

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Anderson et al., *Science* **269**, (1995); Davies et al., *Phys. Rev. Lett* **75**, (1995); Ketterle et al., *Rev. Mod. Phys.* **74**, (2002).

Bose-Einstein Condensation

BEC implies **O**ff-**D**iagonal **L**ong-**R**ange **O**rder (ODLRO).

$$\langle\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}')\rangle \equiv \mathrm{Tr}\{\hat{\rho}\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}')\} \equiv \hat{\rho}_1(\mathbf{r},\mathbf{r}')$$

has an eigenvalue $\approx N$ (Total number of particles), $|\mathbf{r}-\mathbf{r}'| \to \infty$

- $\hat{\rho}_1$: Single-particle density matrix
- $\hat{\rho}$: Density operator of the system

O. Penrose and L. Onsager, *Phys. Rev*, **104**, (1956);
M. Ueda, *Fundamentals of Bose Einstein Condensation*, (2010).

General Criteria for BEC

When interactions are present ⇒ Single-particle energy levels are not defined. Define, reduced single-particle density operator

$$\hat{\rho}_1 \equiv \mathrm{Tr}_{2,3,\cdots N}\, \hat{\rho}$$

where $\mathrm{Tr}_{2,3,\cdots N}$ →Trace of $\hat{\rho}$ w.r.t particles $2, 3, \cdots N$

- Define $\hat{\sigma}_1 = N\hat{\rho}_1$
- **Penrose-Onsager condition**:

$$\frac{n_M}{N} = e^{\mathcal{O}(1)}$$

- n_M: largest eigenvalue of $\hat{\sigma}_1$, condensation occurs in corresponding eigenstate
- $e^{\mathcal{O}(1)}$: positive number of the order of unity.

O. Penrose and L. Onsager, *Phys. Rev*, **104**, (1956)

Gross-Pitaevskii equation

- Equation of motion of the condensate wavefunction is given by Gross-Pitaevskii equation(GPE), strictly valid at $T = 0$K.

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{trap}(\mathbf{r}) + gN|\psi|^2 \right]\psi,$$

  - $\psi \equiv \psi(\mathbf{r}, t)$: condensate wave function
  - $g = \dfrac{4\pi\hbar^2 a}{m}$
  - a: atomic scattering length > 0 : repulsive
  - N: Number of atoms in the condensate

$$V_{trap} = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

E. P. Gross, *Il Nuovo Cimento Series 10*, **20**, (1961);
L. P. Pitaevskii, *Soviet Physics JETP-USSR*, **13**, (1961);
C. Pethick & H. Smith, *Bose-Einstein Condensation in Dilute Gases*, (2008)

Why do we study finite temperature effects?

Region of interest :: $0 < T < T_c$

- $T = 0K$ is physically unattainable. Experiments take place at finite temperatures.
- When $T \neq 0$, the condensate co-exists with the *thermal cloud*. Interactions between condensate and non-condensate(thermal) atoms cannot be neglected.

Many-body Hamiltonian

$$\hat{H} = \int \underbrace{d\mathbf{r}\,\hat{\psi}^\dagger(\mathbf{r},t)\left[\hat{h}(\mathbf{r})-\mu\right]\hat{\psi}(\mathbf{r},t)}_{\text{single-particle part}}$$
$$+\frac{1}{2}\iint \underbrace{d\mathbf{r}d\mathbf{r}'\,\hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}^\dagger(\mathbf{r}',t)U(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}',t)\hat{\psi}(\mathbf{r},t)}_{\text{two-particle interaction term}}$$

where $\hat{h} = K.E + V_{\rm trap}$

$$U(\mathbf{r}-\mathbf{r}') = g\delta(\mathbf{r}-\mathbf{r}'), \left\langle \int d\mathbf{r}\,\hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}(\mathbf{r},t) \right\rangle = N$$

U:: Repulsive contact interaction; N:: Total number of atoms

$$\left[\hat{\psi}(\mathbf{r}),\hat{\psi}(\mathbf{r}')\right] = \left[\hat{\psi}^\dagger(\mathbf{r}),\hat{\psi}^\dagger(\mathbf{r}')\right] = 0; \left[\hat{\psi}(\mathbf{r}),\hat{\psi}^\dagger(\mathbf{r}')\right] = \delta(\mathbf{r}-\mathbf{r}')$$

A.Griffin, *Phys. Rev. B*, **53**, (1996)

Hartree-Fock-Bogoliubov(HFB) approximation

The Bose field operator is

$$\hat{\psi}(\mathbf{r},t) = \sum_{i=0} \hat{\alpha}_i(t)\psi_i(\mathbf{r}) = \hat{\alpha}_0(t)\psi_0(\mathbf{r}) + \sum_{i=1} \hat{\alpha}_i(t)\psi_i(\mathbf{r}),$$

$$\hat{\alpha}_i^\dagger|n_0 n_1,\cdots,n_i,\cdots\rangle = \sqrt{(n_i+1)}|n_0 n_1,\cdots,n_i+1,\cdots\rangle,$$

$$\hat{\alpha}_i|n_0 n_1,\cdots,n_i,\cdots\rangle = \sqrt{n_i}|n_0 n_1,\cdots,n_i-1,\cdots\rangle.$$

BEC occurs when: $n_0 \equiv N_0 \gg 1 \rightarrow N_0, N_0 \pm 1 \approx N_0$
where $N_0 \rightarrow$ Number of **condensate** atoms

HFB approximation: $\hat{\alpha}_0 = \hat{\alpha}_0^\dagger = \sqrt{N_0}$, then

$$\hat{\psi}(\mathbf{r},t) = \sqrt{N_0}\psi_0(\mathbf{r})e^{-i\mu t/\hbar} + \tilde{\psi}(\mathbf{r},t),$$

such that, $\langle\tilde{\psi}(\mathbf{r},t)\rangle = \langle\tilde{\psi}^\dagger(\mathbf{r},t)\rangle = 0.$

Dalfovo et al, *Rev. Mod. Phys.*,**71**, (1999)

Generalized GP equation

Equation of motion of the Bose field operator

$$i\hbar\frac{\partial\hat{\psi}(\mathbf{r},t)}{\partial t} = (\hat{h}-\mu)\hat{\psi}(\mathbf{r},t) + g\hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)\hat{\psi}(\mathbf{r},t)$$

where, $\hat{\psi}(\mathbf{r},t) = \phi(\mathbf{r}) + \tilde{\psi}(\mathbf{r},t)$, and $\phi(\mathbf{r}) = \sqrt{N_0}\psi_0(\mathbf{r})$. $\phi/\tilde{\psi}$ is the condensate/non-condensate part. $U(1)$ gauge symmetry broken.

$$\tilde{\psi}^\dagger(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t) \simeq 2\underbrace{\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle}_{\tilde{n}}\tilde{\psi}(\mathbf{r},t) + \underbrace{\langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle}_{\tilde{m}}\tilde{\psi}^\dagger(\mathbf{r},t)$$

$\tilde{n} \rightarrow$ Non-condensate density; $\tilde{m} \rightarrow$ Anomalous average

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$\tilde{n} \rightarrow$ Non-condensate density; $\tilde{m} \rightarrow$ Anomalous average

3-field correlation term $\langle\tilde{\psi}^{\dagger}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\rangle = 0$.

Generalized GP equation

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Generalized GPE

Including the thermal component and anomalous term, the generalized GP equation is

$$(\hat{h}-\mu)\phi(\mathbf{r})+g|\phi(\mathbf{r})|^2\phi(\mathbf{r})+\underbrace{2g\tilde{n}(\mathbf{r})\phi(\mathbf{r})+g\tilde{m}(\mathbf{r})\phi^*(\mathbf{r})}_{T-dependent}=0$$

- $\hat{h}=K.E.+V_{\rm trap}$
- $\int|\phi(\mathbf{r})|^2\,d\mathbf{r}=1$

A.Griffin, *Phys. Rev. B*, **53**, (1996);
D. A. W. Hutchinson, E. Zaremba, and A. Griffin *Phys. Rev. Lett.*,**78**, (1997)

Bogoliubov de-Gennes equations

Equation of motion of the thermal component

$$\begin{aligned} i\hbar\frac{\partial\tilde{\psi}}{\partial t} &= i\hbar\frac{\partial}{\partial t}(\hat{\psi}-\phi), \\ &= (\hat{h}-\mu)\tilde{\psi}+2gn(\mathbf{r})\tilde{\psi}+gm(\mathbf{r})\tilde{\psi}^{\dagger}, \end{aligned}$$

where, $n(\mathbf{r}) = |\phi(\mathbf{r})|^2 + \tilde{n}(\mathbf{r})$; $m(\mathbf{r}) = \phi^2(\mathbf{r}) + \tilde{m}(\mathbf{r})$;
$\tilde{\psi} = \sum_j \left[u_j\hat{\alpha}_j e^{-iE_j t} - v_j^*\hat{\alpha}_j^{\dagger} e^{iE_j t}\right]$;
$u_j, v_j \Rightarrow$ quasiparticle amplitudes
Bogoliubov de-Gennes equations:

$$\begin{aligned} \mathcal{L}u_j &- gmv_j = E_j u_j \\ \mathcal{L}v_j &- gm^* u_j = -E_j v_j \end{aligned}$$

where $\mathcal{L} = \hat{h} - \mu + 2gn(\mathbf{r})$

Non-condensate density

Density of the thermal component:

$$\langle\tilde{\psi}^\dagger(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle = \tilde{n} = \sum_j \left\{ \left[|u_j|^2 + |v_j|^2\right] \langle\hat{\alpha}_j^\dagger\hat{\alpha}_j\rangle + |v_j|^2 \right\}.$$

and multiplying factor

$$\langle\hat{\alpha}_j^\dagger\hat{\alpha}_j\rangle = \frac{1}{e^{\beta E_j} - 1} \equiv N_0(E_j).$$

is the Bose-Einstein distribution. The anomalous average:

$$\langle\tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r})\rangle = \tilde{m} = -\sum_j u_j v_j^* \left[2\langle\hat{\alpha}_j^\dagger\hat{\alpha}_j\rangle + 1\right],$$

A.Griffin, *Phys. Rev. B*, **53**, (1996)

Summary of steps

I. **Generalized GPE**:

$$(\hat{h} - \mu)\phi(\mathbf{r}) + g|\phi(\mathbf{r})|^2\phi(\mathbf{r}) + 2g\tilde{n}(\mathbf{r})\phi(\mathbf{r}) + g\tilde{m}\phi^*(\mathbf{r}) = 0$$

II. **Bogoliubov de-Gennes equations**:

$$\begin{aligned} \mathcal{L}u_j &- gmv_j = E_j u_j \\ \mathcal{L}v_j &- gm^* u_j = -E_j v_j \end{aligned}$$

where $\mathcal{L} = \hat{h} - \mu + 2g(|\phi(\mathbf{r})|^2 + \tilde{n})$

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where $\mathcal{L} = \hat{h} - \mu + 2g(|\phi(\mathbf{r})|^2 + \tilde{n})$

III. **Non-condensate density**:

$$\begin{aligned} \tilde{n}(\mathbf{r}) &= \sum_j \left\{ \left[|u_j|^2 + |v_j|^2\right] \langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle + |v_j|^2 \right\} \\ \tilde{m}(\mathbf{r}) &= -\sum_j u_j v_j^* [2\langle \hat{\alpha}_j^\dagger \hat{\alpha}_j \rangle + 1] \end{aligned}$$

Summary of steps

I. **Generalized GPE**:

$$(\hat{h}-\mu)\phi(\mathbf{r}) + g|\phi(\mathbf{r})|^2\phi(\mathbf{r}) + 2g\tilde{n}(\mathbf{r})\phi(\mathbf{r}) + g\tilde{m}\phi^*(\mathbf{r}) = 0$$

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Eigenvalue Problem

$$E\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}\mathcal{L} & -gm\\ gm^* & -\mathcal{L}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}$$

where, $u(\mathbf{r})=\sum_{i=0}^{n}c_i\varphi_i(\mathbf{r}), v(\mathbf{r})=\sum_{i=0}^{n}d_i\varphi_i(\mathbf{r})$

$|\varphi_i\rangle$'s are the harmonic-oscillator eigenstates.

$$E\begin{pmatrix}c_0\\ \vdots\\ c_n\\ d_0\\ \vdots\\ d_n\end{pmatrix}=\begin{pmatrix}\mathcal{L}_{00} & \cdots & \mathcal{L}_{0n} & -\mathcal{B}_{00} & \cdots & -\mathcal{B}_{0n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \mathcal{L}_{n0} & \cdots & \mathcal{L}_{nn} & -\mathcal{B}_{n0} & \cdots & -\mathcal{B}_{nn}\\ \mathcal{B}_{00} & \cdots & \mathcal{B}_{0n} & -\mathcal{L}_{00} & \cdots & -\mathcal{L}_{0n}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \mathcal{B}_{n0} & \cdots & \mathcal{B}_{nn} & -\mathcal{L}_{n0} & \cdots & -\mathcal{L}_{nn}\end{pmatrix}\begin{pmatrix}c_0\\ \vdots\\ c_n\\ d_0\\ \vdots\\ d_n\end{pmatrix}$$

Pu et. al, *Phys. Rev. Lett.*, **80**, (1998); Reidl et. al, *Phys. Rev. A*, **59**, (1999)

Problems in HFB

- HFB theory is not *gapless*. Violates Hugenholtz-Pines theorem.
  Reason :: Approximate factorization of operator averages
- The anomalous pair average $\tilde{m}$ is divergent.
  Reason :: Inconsistent treatment of collisions through contact potential.Treats collisions of different energy with same probability.

Gapless finite temperature approximation

$\Downarrow$

Neglect $\tilde{m}$.

$\Downarrow$

HFB-Popov approximation

Valid in $0 < T \lesssim 0.5 T_c$

N. P. Proukakis and B. Jackson, *J. Phys. B*, **41**, (2008)

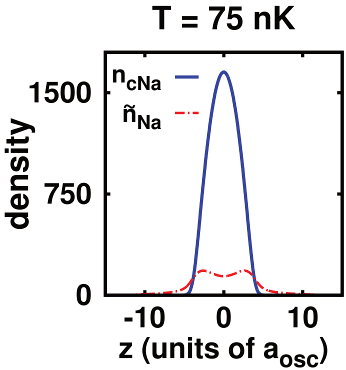
Preliminary Results for ^{23}Na condensate

Quasi-1d geometry
Cigar-shaped trap

Trapping potential:
$\omega_{\perp(\mathrm{Na})} = 2\pi \times 40.2\,\mathrm{Hz}, \omega_{z(\mathrm{Na})} = 2\pi \times 4.55\,\mathrm{Hz}$

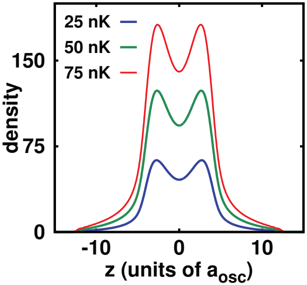
Scattering length:
$\mathrm{a_{Na}} = 2.75\,\mathrm{nm}$

Total number of atoms:
$N_{\mathrm{T}} = 10\,000$



The noncondensate (dashed) and the condensate (solid) densities at $T = 75\,\mathrm{nK}$

Noncondensate density for 10 000 Sodium atoms at various temperatures



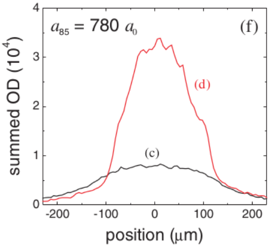
Unique feature of binary BEC

Role of interactions

Phase Separation

$$U_{11}U_{22} - (U_{12})^2 > 0$$

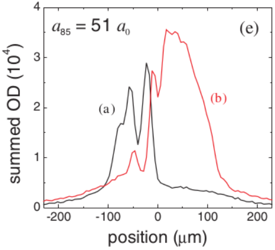
Miscible regime



^{85}Rb–^{87}Rb

$$U_{11}U_{22} - (U_{12})^2 < 0$$

Immiscible regime



Papp et. al, *Phys. Rev. Lett.*,**101**, (2008)

Experimental realization of binary BEC

2 different **atoms**

- ^{87}Rb–^{41}K
  Thalhammer et. al, PRL, **100**, (2008)
- ^{84}Sr–^{87}Rb
  Pasquiou et. al, arXiv:1305.5935, (2013)
- ^{23}Na–^{87}Rb
  Xiong et. al, arXiv:1305.7091, (2013)
- ^{133}Cs–^{87}Rb
  McCarron et. al, PRA(R), **84**, (2011)

2 different **isotopes**

- ^{85}Rb–^{87}Rb
  Papp et. al, PRL, **101**, (2008)

2 different **hyperfine states**

- $|F=1, m_F=+1\rangle$, $|F=2, m_F=-1\rangle$ of ^{87}Rb
  Tojo et. al, PRA, **82**, (2010)

Dynamical evolution

Instabilities

Instabilities in phase separated regime

- Rayleigh-Taylor instability
- Kelvin-Helmholtz instability

Sasaki et al. , *Phys. Rev. A* **80**, (2009);
S. Gautam and D. Angom , *Phys. Rev. A* **81**, (2010);
Takeuchi et. al , *Phys. Rev. B* **81**, (2010);
Kadokura et al. , *Phys. Rev. A* **85**, (2012);
AR, S. Gautam, D. Angom, arXiv:1210.0381, (2012)

Hamiltonian of binary BEC

$$\begin{aligned} H &= \sum_{k=1,2} \int d\mathbf{r} \hat{\Psi}_k^\dagger(\mathbf{r},t) \Big[-\frac{\hbar^2 \nabla^2}{2m_k} + V_k(\mathbf{r}) - \mu_k \\ &+ \frac{U_{kk}}{2} \hat{\Psi}_k^\dagger(\mathbf{r},t) \hat{\Psi}_k(\mathbf{r},t) \Big] \hat{\Psi}_k(\mathbf{r},t) \\ &+ U_{12} \int d\mathbf{r} \hat{\Psi}_1^\dagger(\mathbf{r},t) \hat{\Psi}_2^\dagger(\mathbf{r},t) \hat{\Psi}_1(\mathbf{r},t) \hat{\Psi}_2(\mathbf{r},t). \end{aligned}$$

Equation of motion for the Bose field operators

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \hat{\Psi}_1 \\ \hat{\Psi}_2 \end{pmatrix} = \begin{pmatrix} \hat{h}_1 + U_{11} \hat{\Psi}_1^\dagger \hat{\Psi}_1 & U_{12} \hat{\Psi}_2^\dagger \hat{\Psi}_1 \\ U_{21} \hat{\Psi}_1^\dagger \hat{\Psi}_2 & \hat{h}_2 + U_{22} \hat{\Psi}_2^\dagger \hat{\Psi}_2 \end{pmatrix} \begin{pmatrix} \hat{\Psi}_1 \\ \hat{\Psi}_2 \end{pmatrix}$$

where $\hat{h}_k = -\dfrac{\hbar^2 \nabla^2}{2m_k} + V_k(\mathbf{r}) - \mu_k$

Coupled Generalized GP equation

From **HFB-Popov** approximation

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + \begin{pmatrix}\tilde{\psi}_1\\ \tilde{\psi}_2\end{pmatrix},$$

$$\hat{h}_1\phi_1 + U_{11}\left[n_{1c} + 2\tilde{n}_1\right]\phi_1 + U_{12}[n_{2c} + \tilde{n}_2]\phi_1 = 0,$$

$$\hat{h}_2\phi_2 + U_{22}\left[n_{2c} + 2\tilde{n}_2\right]\phi_2 + U_{12}[n_{1c} + \tilde{n}_1]\phi_2 = 0.$$

- $n_{kc} = |\phi_k|^2$: Condensate density of k^{th} species
- $\tilde{n}_k$: Non-condensate density of k^{th} species

BdG equations

Bogoliubov Transformation

$$\begin{aligned}\tilde{\psi}_k(\mathbf{r}, t) &= \sum_j \left[u_{kj}(\mathbf{r})\hat{\alpha}_j(\mathbf{r})e^{-iE_j t} - v^*_{kj}(\mathbf{r})\hat{\alpha}_j(\mathbf{r})^\dagger e^{iE_j t}\right], \\ \tilde{\psi}^\dagger_k(\mathbf{r}, t) &= \sum_j \left[u_{kj}(\mathbf{r})^*\hat{\alpha}^\dagger_j(\mathbf{r})e^{iE_j t} - v_{kj}(\mathbf{r})\hat{\alpha}_j(\mathbf{r})e^{-iE_j t}\right]\end{aligned}$$

$k = 1, 2$ is the species index, and

$$[\hat{\alpha}_i, \hat{\alpha}^\dagger_j] = \delta_{ij}; [\hat{\alpha}_i, \hat{\alpha}_j] = 0; [\hat{\alpha}^\dagger_i, \hat{\alpha}^\dagger_j] = 0.$$

BdG Equations

$$
\begin{aligned}
\hat{\mathcal{L}}_1 u_{1j} - U_{11}\phi_1^2 v_{1j} + U_{12}\phi_1(\phi_2^* u_{2j} - \phi_2 v_{2j}) &= \hbar E_j u_{1j}, \\
-\hat{\mathcal{L}}_1 v_{1j} + U_{11}\phi_1^{*2} u_{1j} - U_{12}\phi_1^*(\phi_2 v_{2j} - \phi_2^* u_{2j}) &= \hbar E_j v_{1j}, \\
\hat{\mathcal{L}}_2 u_{2j} - U_{22}\phi_2^2 v_{2j} + U_{12}\phi_2(\phi_1^* u_{1j} - \phi_1 v_{1j}) &= \hbar E_j u_{2j}, \\
-\hat{\mathcal{L}}_2 v_{2j} + U_{22}\phi_2^{*2} u_{2j} - U_{12}\phi_2^*(\phi_1 v_{1j} - \phi_1^* u_{1j}) &= \hbar E_j v_{2j},
\end{aligned}
$$

$$
\begin{aligned}
\text{where, } \hat{\mathcal{L}}_1 &= \left(\hat{h}_1 + 2U_{11}n_1 + U_{12}n_2\right), \\
\hat{\mathcal{L}}_2 &= \left(\hat{h}_2 + 2U_{22}n_2 + U_{12}n_1\right), \\
n_1 &= n_{1c} + \tilde{n}_1, n_2 = n_{2c} + \tilde{n}_2.
\end{aligned}
$$

Non-Condensate density:

$$
\tilde{n}_k = \sum_j \left\{ \left[|u_{kj}|^2 + |v_{kj}|^2 \right] N_0(E_j) + |v_{kj}|^2 \right\}
$$

BdG Equations

$$
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where,

$$
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Non-Condensate density:

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\tilde{n}_k = \sum_j \left\{ \left[|u_{kj}|^2 + |v_{kj}|^2\right] N_0(E_j) + |v_{kj}|^2 \right\}
$$

Solving non-condensate density $\tilde{n}$

- We numerically solve the set of coupled GGPE at $T = 0$ to obtain the condensate wavefunction of both the species using Crank-Nicolson method. In this case the density of thermal component $\tilde{n}_k = 0$.
- Using this condensate wavefunction, we cast the coupled Bogoliubov de-Gennes equations in matrix form in basis of harmonic oscillator eigenstates.
- We then numerically diagonalize the matrix using `LAPACK` subroutine to calculate the energy eigenvalues E_j's and eigenstates u_j, v_j's.

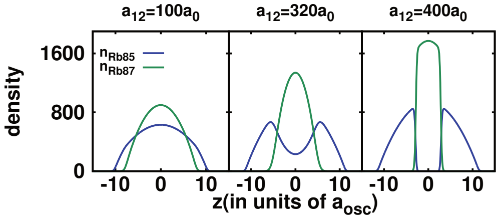
P. Muruganandam and S. K. Adhikari *Comp. Phys. Comm.* **180**, (2009);
S. Gautam and D. Angom *J. Phys. B* **43**, (2010);
S. Gautam and D. Angom *J. Phys. B* **44**,(2011);
Anderson et. al, *LAPACK Users' Guide*, **3rd** Edn., (1999)

Solving non-condensate density $\tilde{n}$

- Furthermore, we ensure that the eigenvalues E_j's are real and two zero energy modes are obtained corresponding to each condensate. Imaginary eigenvalues will correspond to metastable state.
- We finally calculate the density of the thermal component $\tilde{n}_k$ by summing over all positive, real eigenvalues except the two zero-energy or Goldstone modes.
- We iterate the above procedure to self-consistency.

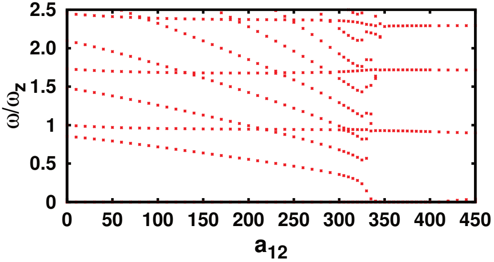
A.Griffin, *Phys. Rev. B*, **53**, (1996);
D. A. W. Hutchinson, E. Zaremba, and A. Griffin *Phys. Rev. Lett.*,**78**, (1997)

Mode evolution in highly phase-separated condensate



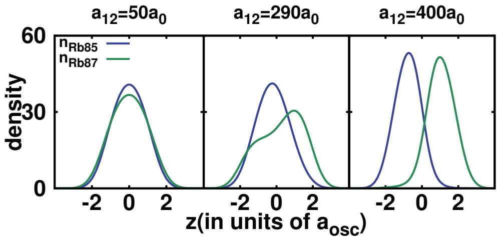
$$\mathrm{a}_{^{85}\mathrm{Rb}}/\mathrm{a}_{^{87}\mathrm{Rb}} = 280\mathrm{a}_0/99\mathrm{a}_0$$
$$\mathrm{N}_{^{85}\mathrm{Rb}} = \mathrm{N}_{^{87}\mathrm{Rb}} = 10\,000$$
$$\omega_{z^{85}\mathrm{Rb}} = 2\pi \times 4.55\,\mathrm{Hz}$$
$$\omega_{z^{87}\mathrm{Rb}} = 2\pi \times 3.89\,\mathrm{Hz}$$
$$\lambda_{^{85}\mathrm{Rb}} = 8.835, \lambda_{^{87}\mathrm{Rb}} = 8.277$$

Energy excitation spectra with varying $\mathrm{a}_{^{85}\mathrm{Rb}^{87}\mathrm{Rb}}$



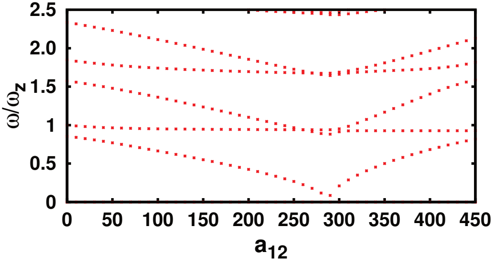
- One of the excited mode goes to zero with increasing a_{12},
- Disappearance of certain modes, unique to symmetry preserving solutions.

Symmetry breaking at $T = 0$



$a_{^{85}\mathrm{Rb}} : a_{^{87}\mathrm{Rb}} = 1.01$
$N_{^{85}\mathrm{Rb}} = N_{^{87}\mathrm{Rb}} = 100$
$\omega_{z\,^{85}\mathrm{Rb}} = 2\pi \times 4.55\,\mathrm{Hz}$
$\omega_{z\,^{87}\mathrm{Rb}} = 2\pi \times 3.89\,\mathrm{Hz}$
$\lambda = \omega_\perp/\omega_z = 14.0$

Energy excitation spectra with varying a_{12}

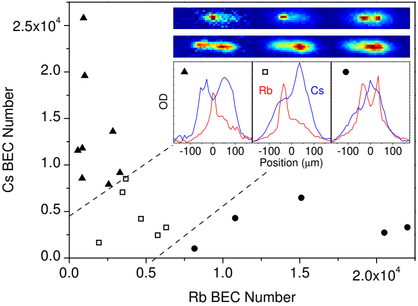


- One of the excited mode goes soft at some critical value of a_{12}.
- The modes do not disappear. It is a characteristic of symmetry breaking solution.

Experimental Realization of ^{133}Cs–^{87}Rb mixture

Quasi-1d geometry
Cigar-shaped trap

Trapping potential:
$(\omega_\perp, \omega_z)_{\rm Cs} = 2\pi \times (40.2, 4.55)$ Hz
$(\omega_\perp, \omega_z)_{\rm Rb} = 2\pi \times (32.2, 3.89)$ Hz
Scattering lengths:
$a_{\rm Cs} = 280a_0, a_{\rm Rb} = 100a_0,$
$a_{\rm CsRb} = 650a_0$
Temperature:
$T = 15{\rm nK}$

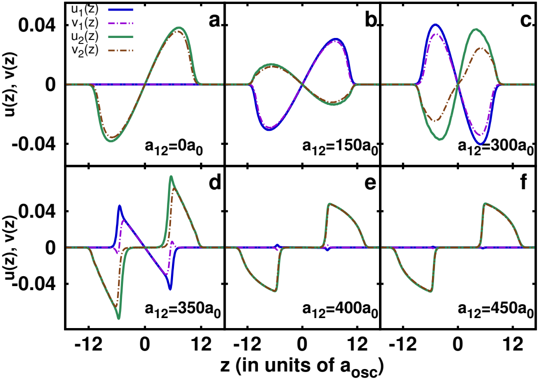


McCarron et. al, *Phys. Rev. A (R)* **84**, (2011)

Calculational Details:

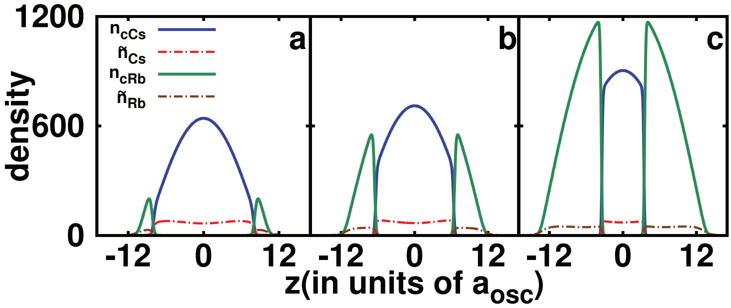
- For the given set of experimental parameters, the ground state is symmetric.
- At lower interspecies interaction a_{12}, only 2 Goldstone modes are present.
- With increasing a_{12}, the first excited mode gets damped and becomes degenerate with the ground state.
- We obtain an extra zero energy or Goldstone mode, in addition to the 2 Goldstone modes corresponding to each of the condensate formed due to $U(1)$ gauge symmetry breaking.

Metamorphosis: From Bogolon to Goldstone



To calculate the density of thermal component $\tilde{n}_k$, we leave the 3 zero energy modes.

Condensate & noncondensate density at $T = 15\,\mathrm{nK}$



$N_{\mathrm{Cs}} = 8570$, $N_{\mathrm{Rb}} = 840$

$N_{\mathrm{Cs}} = 8510$, $N_{\mathrm{Rb}} = 3680$

$N_{\mathrm{Cs}} = 6470$, $N_{\mathrm{Rb}} = 15100$

Conclusion

- We have generalized HFB-Popov approximation to analyze finite temperature effects on binary mixtures of Bose condensed gases.
- Symmetry preserving solution of highly phase separated condensates give rise to a third additional Goldstone mode. In a single species BEC, interspecies interaction is absent.
- In symmetry breaking solution of TBEC, one of the excited mode goes soft which gives rise to an additional Goldstone mode which is unique to binary BEC.
- Finally, we have obtained the equilibrium solutions of the recent experimental realization of ^{133}Cs–^{87}Rb binary mixture at finite temperature.($T \neq 0$, $T < T_c$)

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THANK YOU